Feedback Control of Dynamic Systems

Yves Briere

yves.briere@isae.fr

I. Introduction

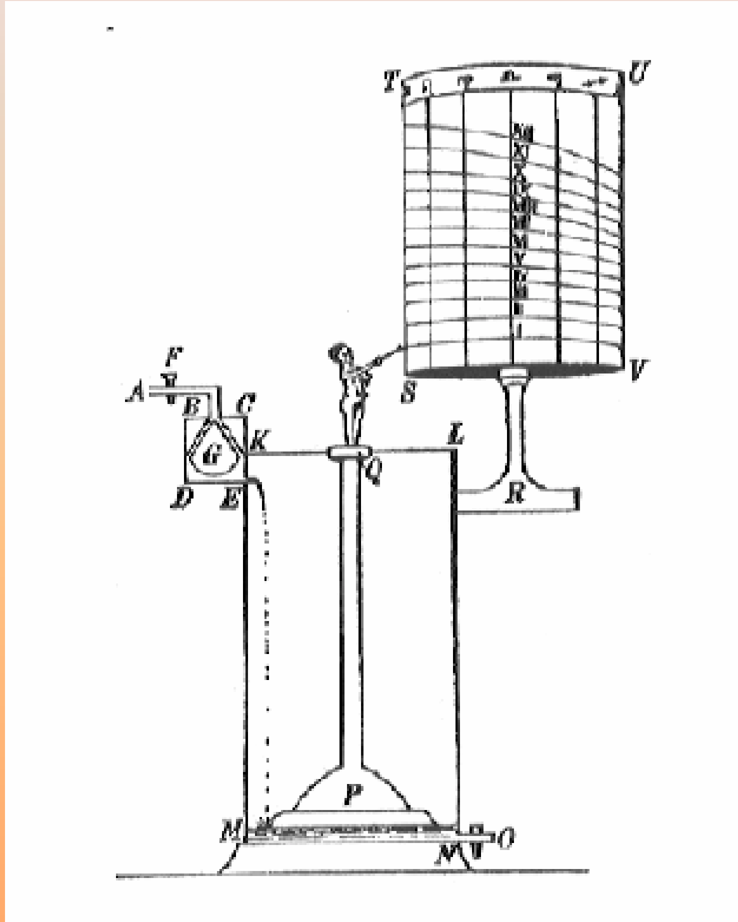
Introduction

- Aim of the course
  - Give a general overview of classical and modern control theory
  - Give a general overview of modern control tools
- Prerequisites
  - Mathematics : complex numbers, linear algebra

Introduction

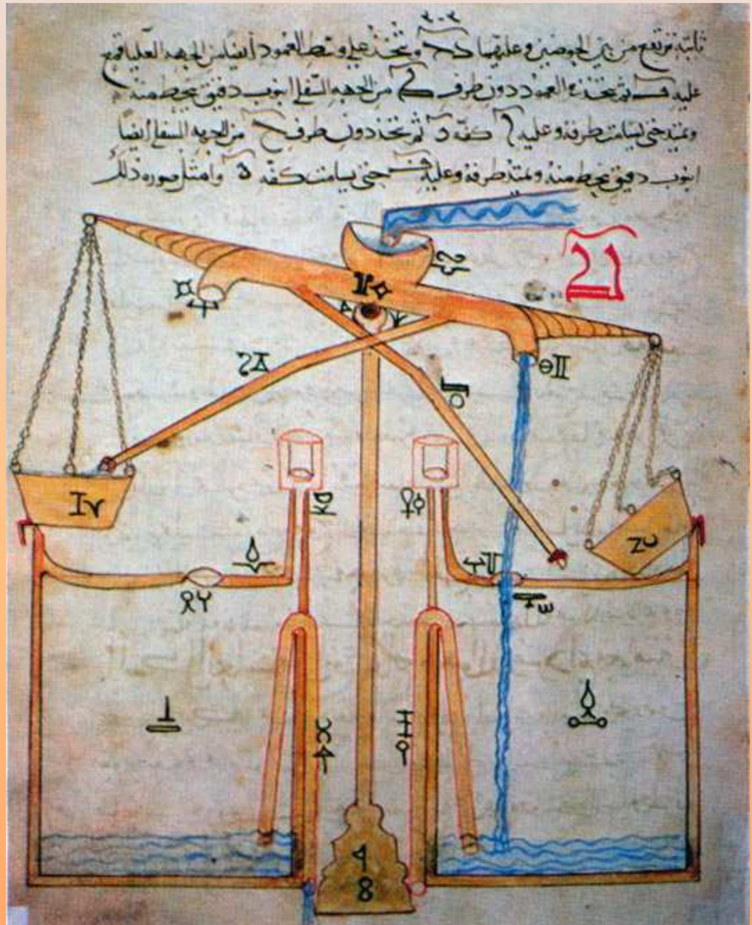
- Tools
  - Matlab / Simulink
- Book
  - « Feedback Control of Dynamics Systems », *Franklin, Powell, Amami-Naeini*, Addison-Wessley Pub Co
  - Many many books, websites and free references...

Introduction



270 BC : the clepsydra and other hydraulically regulated devices for time measurement (Ktesibios)

Introduction



1136-1206 : Ibn al-Razzaz al-Jazari

“The Book of Knowledge of Ingenious Mechanical Devices”
→ crank mechanism, connecting rod, programmable automaton, humanoid robot, reciprocating piston engine, suction pipe, suction pump, double-acting pump, valve, combination lock, cam, camshaft, segmental gear, the first mechanical clocks driven by water and weights, and especially the crankshaft, which is considered the most important mechanical invention in history after the wheel

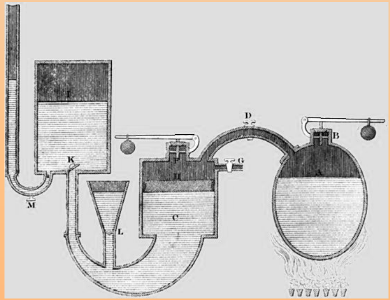
Introduction

1600-1900 : pre-industrial revolution

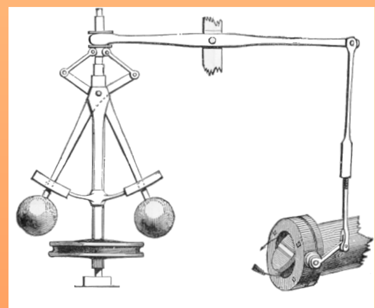
Thermostatic regulators (Cornelius Drebbel 1572 - 1633)

Water level regulation (flush toilet, steam machine)

Steam engine pressure regulation (D. Papin 1707)



Windmill speed regulation. 1588 : mill hoper ; 1745 : **fantail** by Lee ; 1780 : speed regulation by Mead



Centrifugal mechanical governor (James Watt, 1788)

Introduction

1800-1935 : mathematics, basis for control theory

Differential equations → first analysis and proofs of stability condition for feedback systems (Lagrange, Hamilton, Poncelet, Airy-1840, Hermite-1854, Maxwell-1868, Routh-1877, Vyshnegradsky-1877, Hurwitz-1895, Lyapunov-1892)

Frequency domain approach (Minorsky-1922, Black-1927, Nyquist-1932, Hazen-1934)

1940-1960 : classical period

Frequency domain theory : (Hall-1940, Nichols-194, Bode-1938)

Stochastic approach (Kolmogorov-1941, Wiener and Bigelow-1942)

Information theory (Shannon-1948) and **cybernetics** (Wiener-1949)

Introduction

1960-1980 : modern period, aeronautics and spatial industry

Non linear and time varying problems (Hamel-1949, Tsypkin-1955, Popov-1961, Yakubovich-1962, Sandberg-1964, Narendra-1964, Desoer-1965,Zames-1966)

Optimal control and Estimation theory (Bellman-1957, Pontryagin-1958,Kalman-1960)

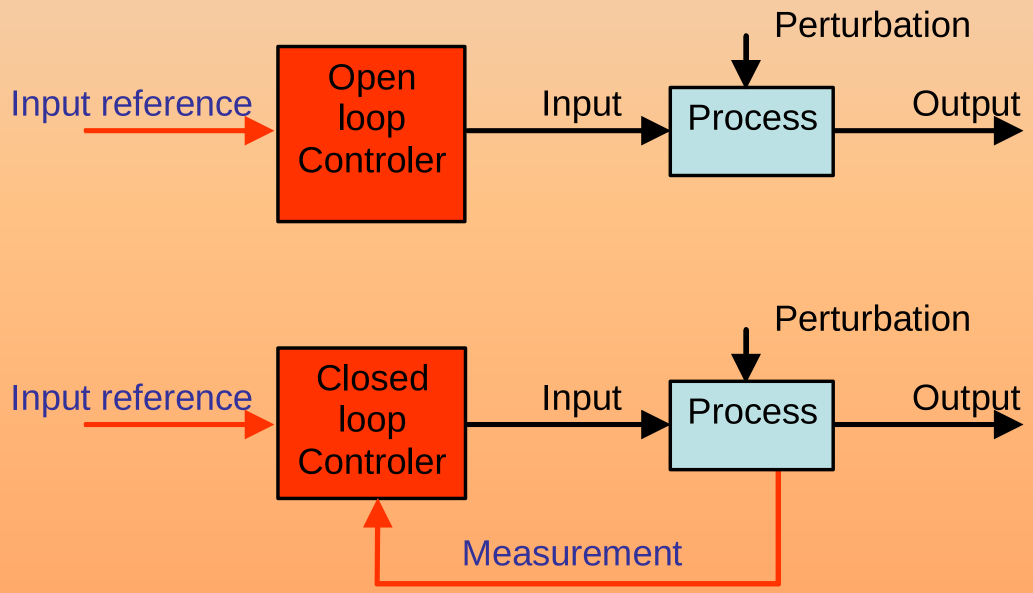
Control by computer, discrete systems theory : (Shannon-1950, Jury-1960, Ragazzini and Zadeh-1952, Ragazzini and Franklin-1958,(Kuo-1963, Aström-1970)

1980-... : simulation, computers, etc...

Introduction

What is automatic control ?

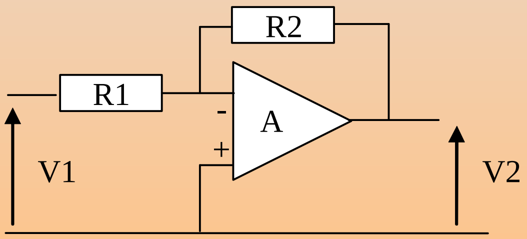
→ Basic idea is to enhance open loop control with feedback control
 → This seemingly idea is tremendously powerfull
 → Feedback is a key idea in control



Introduction

Example : the feedback amplifier

Harold Black, 1927



Amplifier A has a high gain (say 40dB)

$$\frac{V2}{V1} = -\frac{R2}{R1} \cdot \frac{1}{1+\frac{1}{A}\left(1+\frac{R2}{R1}\right)} \approx -\frac{R2}{R1}$$

Resulting gain is determined by passive components !

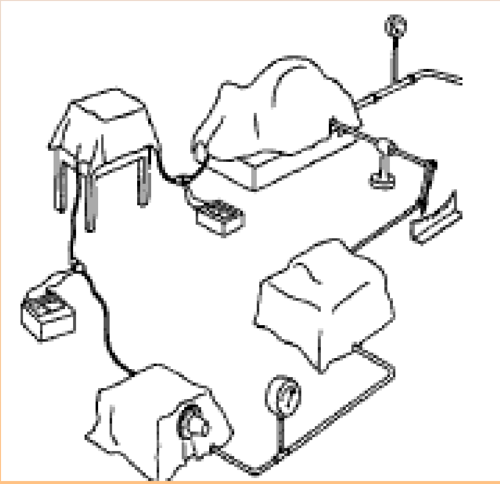
→ amplification is linear

→ reduced delay

→ noise reduction

Introduction

Use of block diagrams

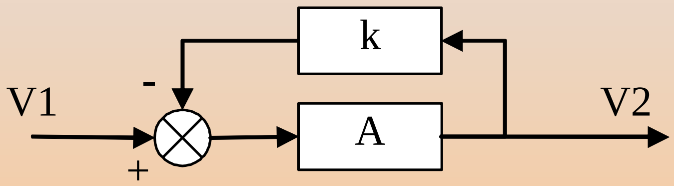


→ Capture the essence of behaviour
→ standard drawing
→ abstraction
→ information hiding
→ points similarities between systems

Same tools for :
→ generation and transmission of energy
→ transmission of informaiton
→ transportation (cars, aerospace, etc...)
→ industrial processes, manufacturing
→ mechatronics, instrumentation
→ Biology, medicine, finance, economy...

Introduction

Basic properties of feedback (1)

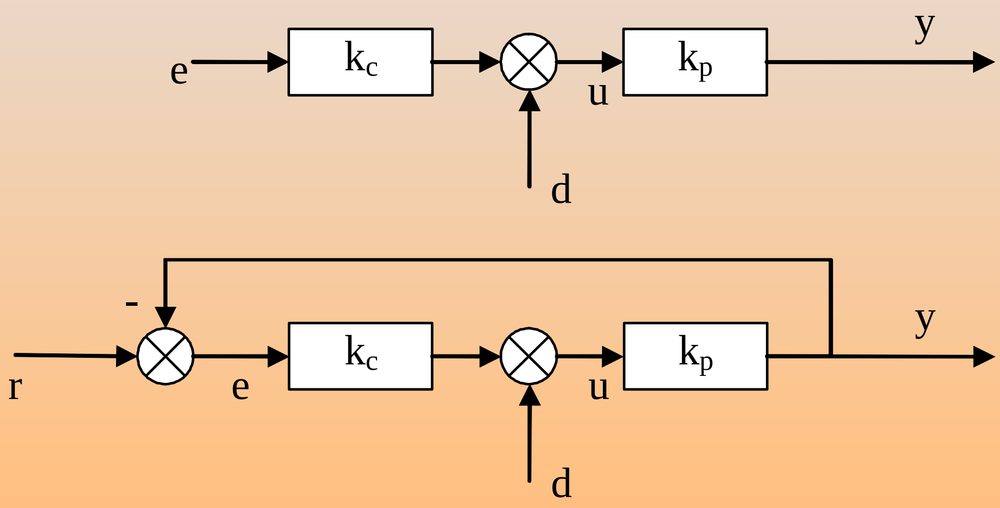


$$V2 = A \cdot (V1 - k \cdot V2)$$
$$V2 \cdot (1 + A \cdot k) = A \cdot V1$$
$$\frac{V2}{V1} = \frac{1}{k} \cdot \frac{1}{1 + \frac{1}{A \cdot k}} \approx \frac{1}{k}$$

→ Resulting gain is determined by feedback !

Introduction

Basic properties of feedback (2) : static properties



r : reference
e : error
d : disturbance
y : output
k_c : control gain
K_p : process gain

Open loop control : $y = k_p \cdot k_c \cdot e + k_p \cdot d$

Closed loop control : $y = \dfrac{k_p \cdot k_c}{1 + k_p \cdot k_c} \cdot r + \dfrac{k_p}{1 + k_p \cdot k_c} \cdot d$

→ If k_c is big enough y tend to r and d is rejected

Introduction

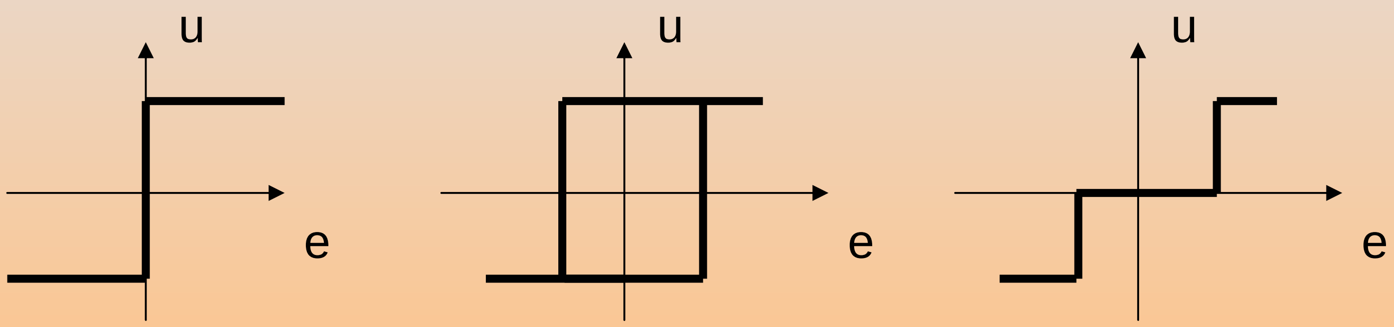
Basic properties of feedback (2) : dynamics properties

Closed loop control can :

→ enhance system dynamics
→ stabilize an unstable system
→ make unstable a stable system ! ☹

Introduction

The On-Off or bang-bang controller : $u = \{u_{max} , u_{min}\}$



The proportional controller : $u=k_c.(r - y)$

Introduction

The proportional **derivative** controller

$$u(t) = k_p \cdot e(t) + k_d \cdot \frac{de(t)}{dt}$$

Gives an idea of future : phase advance

The proportional **integral** controller

$$u(t) = k_p \cdot e(t) + k_i \cdot \int_0^t e(\tau) \cdot d\tau$$

e(t) tends to zero !

II. A first controller design

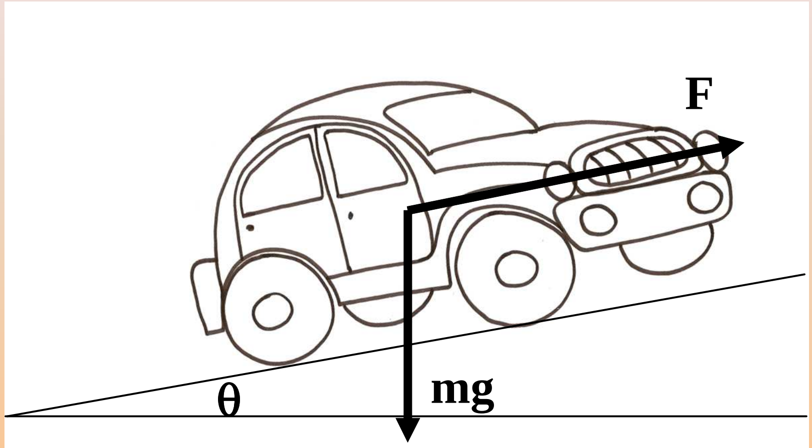
A first control design

- Use of block diagrams
- Compare feedback and feedforward control
- Insight feedback properties :
  - Reduce effect of disturbances
  - Make system insensitive to variations
  - Stabilize unstable system
  - Create well defined relationship between output and reference
  - Risk of unstability
- PID controler : $u(t) = k_p \cdot e(t) + k_d \cdot \frac{de(t)}{dt} + k_i \cdot \int_0^t e(\tau) \cdot d\tau$

Cruise control

A cruise control problem :

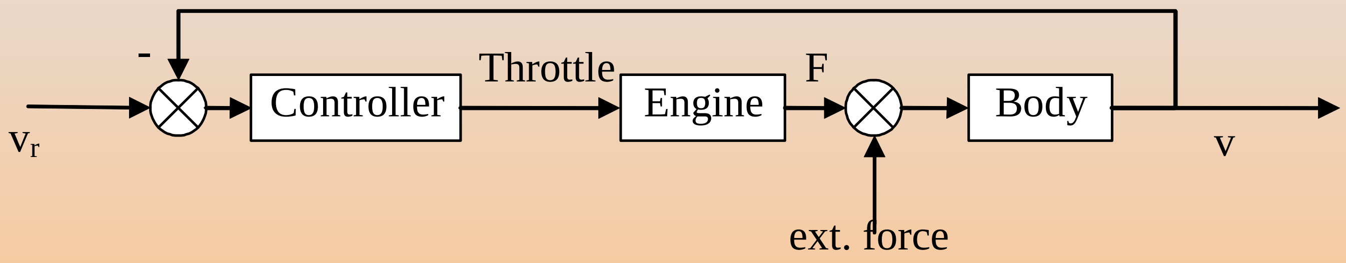
- Process input : gas pedal u
- Process output : velocity v
- Reference : desired velocity v_r
- Disturbance : slope θ



Construct a block diagram

- Understand how the system works
- Identify the major components and the relevant signals
- Key questions are :
  - Where is the essential dynamics ?
  - What are the appropriate abstractions ?
- Describe the dynamics of the blocks

Cruise control



We made the assumptions :

- Essential dynamics relates velocity to force
- The force respond instantly to a change in the throttle
- Relations are linear

We can now draw the process equations

Cruise control

Process linear equations :

$$m \cdot \frac{dv(t)}{dt} + k \cdot v = F - m \cdot g \cdot \theta$$

Reasonable parameters according to experience :

$$\frac{dv(t)}{dt} + 0.02 \cdot v = u - 10 \cdot \theta$$

Where :

- v in $m.s^{-1}$
- u : normalized throttle $0 < u < 1$
- θ slope in rad

Cruise control

Process linear equations :

$$\frac{dv(t)}{dt} + 0.02 \cdot v = u - 10 \cdot \theta$$

PI controller :

$$u(t) = k \cdot (v_r - v(t)) + k_i \cdot \int_0^t (v_r - v(\tau)) \cdot d\tau$$

Combining equations leads to :

$$\frac{d^2 e(t)}{dt^2} + (0.02 + k) \cdot \frac{de(t)}{dt} + k_i \cdot e(t) = 10 \cdot \frac{d\theta(t)}{dt}$$

Integral action

Steady state and $\theta = 0 \rightarrow e = 0$!

Cruise control

Now we can tune k and k_i in order to achieve a given dynamics

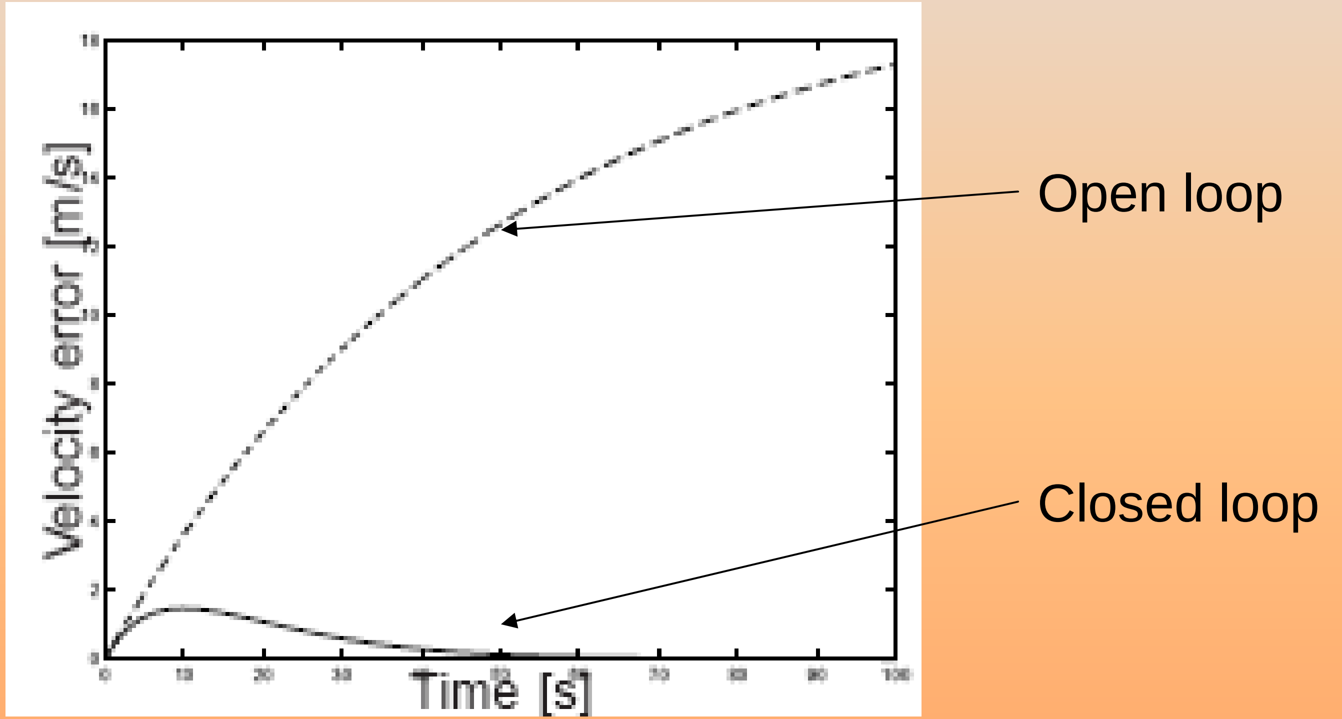
$$\frac{d^2 e(t)}{dt} + (0.02 + k) \cdot \frac{de(t)}{dt} + k_i \cdot e(t) = 10 \cdot \frac{d\theta(t)}{dt}$$

$$\frac{d^2 x(t)}{dt} + 2 \cdot \sigma \cdot \omega_0 \cdot \frac{dx(t)}{dt} + \omega_0{}^2 \cdot x(t) = 0$$

How to choose ω_0 and σ ?

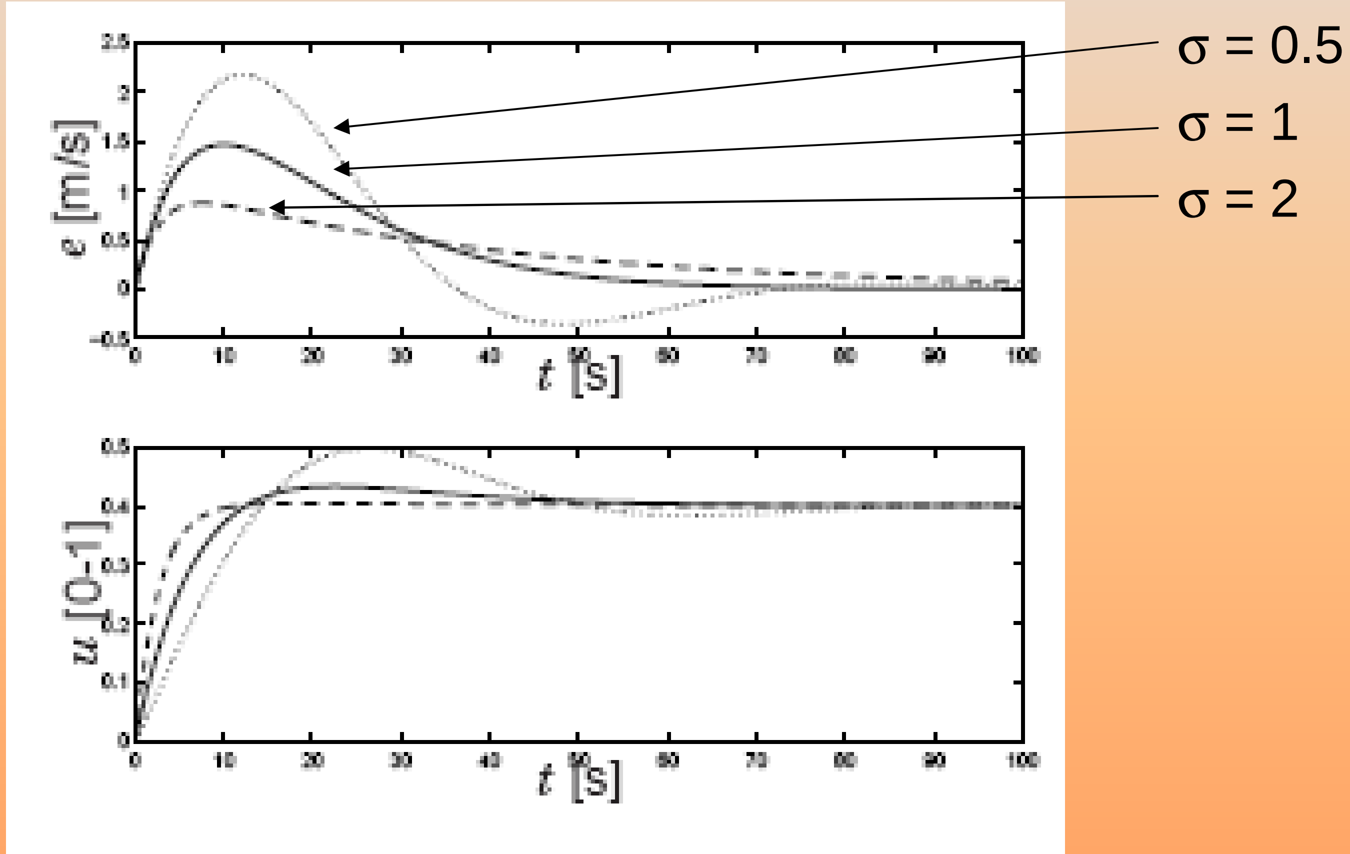
Cruise control

Compare open loop and closed loop



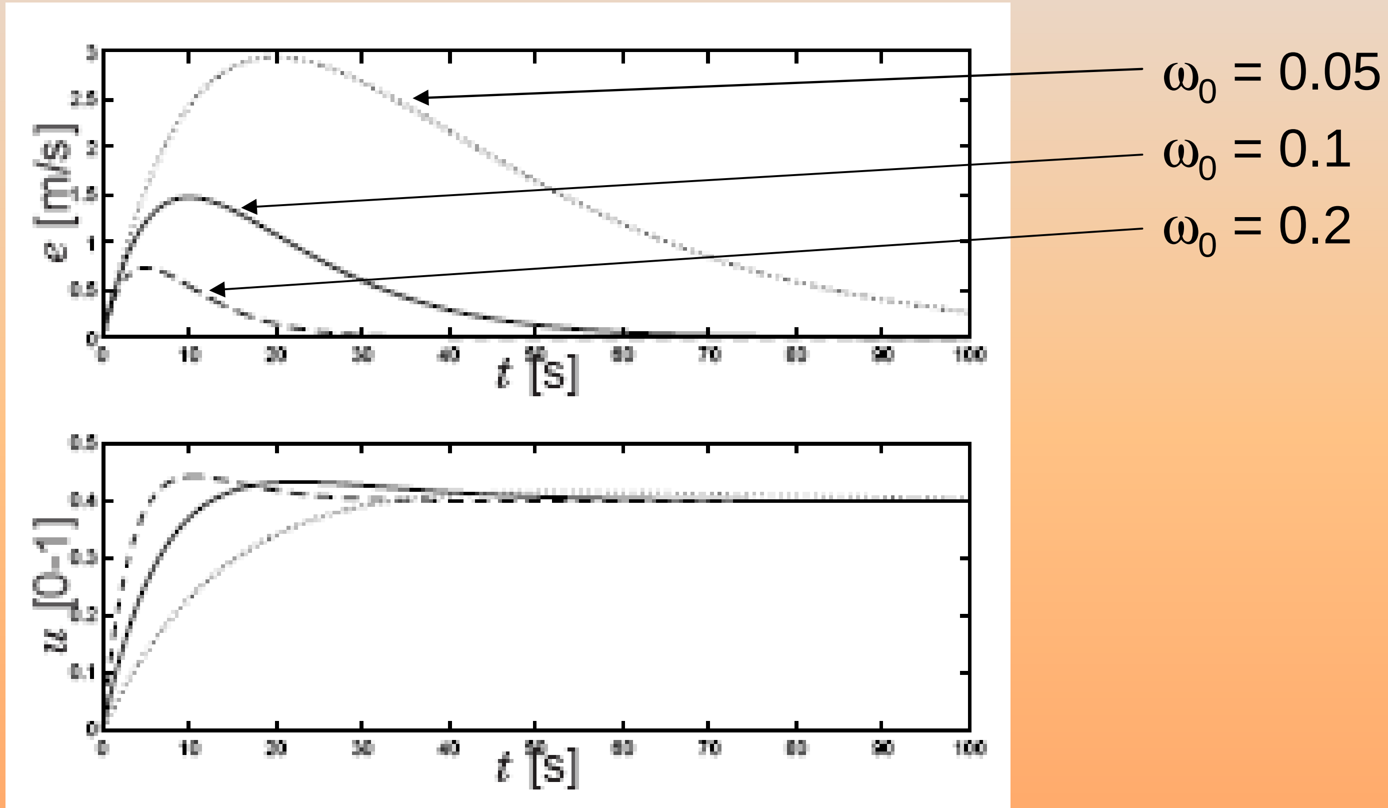
Cruise control

Compare different damping σ (ω_0 = 0.1)



Cruise control

Compare different natural frequencies ω_0 (σ= 1)



Cruise control

Control tools and methods help to :

- Derive equations from the system
- Manipulate the equations
- Understand the equations (standard model)
  - Qualitative understanding concepts
  - Insight
  - Standard form
  - Computations
- **Find controller parameters**
- Validate the results by simulation

END 1

Standard models

Standard models are foundations of the "control language"

Important to :

→ Learn to deal with standard models

→ Transform problems to standard model

The standard model deals with Linear Time Invariant process (LTI), modelized with Ordinary Differential Equations (ODE) :

$$\frac{d^n y(t)}{dt^n} + a_1 \cdot \frac{d^{n-1} y(t)}{dt^{n-1}} ... + a_n \cdot y(t) = b_1 \cdot \frac{d^{n-1} u(t)}{dt^{n-1}} ... + b_n \cdot u(t)$$

Standard models

Example (fundamental) : the first order equation

$$\frac{dy(t)}{dt} + a \cdot y(t) = 0$$

$$\Rightarrow y(t) = y(0) \cdot e^{-a \cdot t}$$

$$\frac{dy(t)}{dt} + a \cdot y(t) = b \cdot u(t)$$

$$\Rightarrow y(t) = y(0) \cdot e^{-a \cdot t} + b \cdot \int_0^t e^{-a \cdot (t-\tau)} \cdot u(\tau) \cdot d\tau$$

Input signal

Initial conditions

Standard models

A higher degree model is not so different :

$$\frac{d^n y(t)}{dt^n} + a_1 \cdot \frac{d^{n-1} y(t)}{dt^{n-1}} ... + a_n \cdot y(t) = 0$$

Characteristic polynomial is :

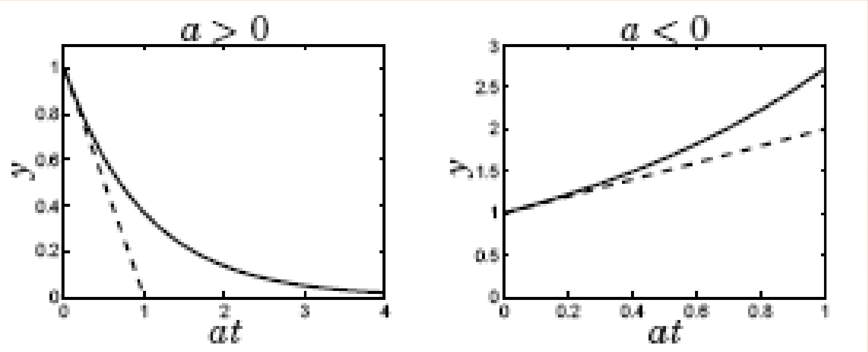
$$A(s) = s^n + a_1 \cdot s^{n-1} ... + a_n$$

If polynomial has n distinct roots α_k then the time solution is :

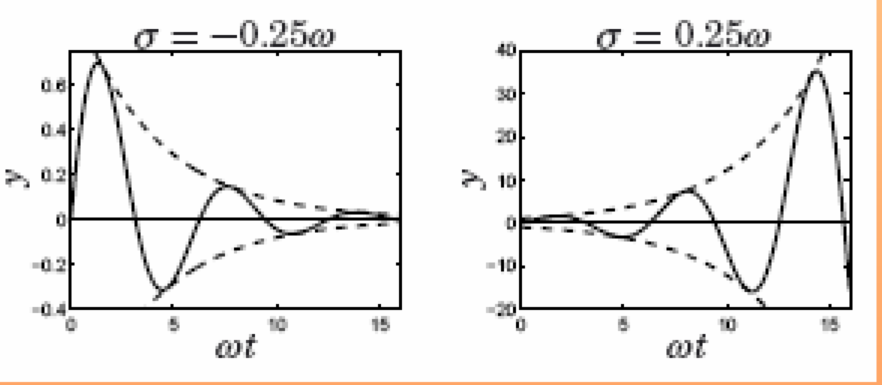
$$y(t) = \sum_{k=1}^{n} C_k \cdot e^{\alpha_k \cdot t}$$

Standard models

Real α_k roots gives first order responses :



Complex $\alpha_k = \sigma \pm i.\omega$ roots gives second order responses :



Standard models

General case (input u) :

$$\frac{d^n y(t)}{dt^n} + a_1 \cdot \frac{d^{n-1} y(t)}{dt^{n-1}} ... + a_n \cdot y(t) = b_1 \cdot \frac{d^{n-1} u(t)}{dt^{n-1}} ... + b_n \cdot u(t)$$

$$\Rightarrow y(t) = \sum_{k=1}^{n} C_k(t) \cdot e^{\alpha_k \cdot t} + \int_0^t g(t-\tau) \cdot d\tau$$

Where :

- $C_k(t)$ are polynomials of t
- $g(t) = \sum_{k=1}^{n} C'_k(t) \cdot e^{\alpha_k \cdot t}$

A system is stable if all poles have negative real parts

Standard models

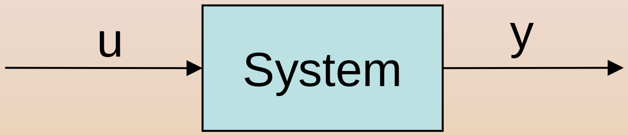
Transfer function

→ without knowing anything about Laplace transform it can be useful to store a_k and b_k coefficients in a convenient way, the transfer function :

$$F(s) = \frac{B(s)}{A(s)} = \frac{s^n + a_1 \cdot s^{n-1} ... + a_n}{b_1 \cdot s^{n-1} ... + b_n}$$

III. The Laplace transform

Laplace transform (1) : convolution



- We assume the system to be LINEAR and TIME INVARIANT

⇨ The output (y) of the the system is related to the input (u) by the convolution :

$$y(t) = \int_{-\infty}^{+\infty} u(\tau) \cdot h(t-\tau) \cdot d\tau$$

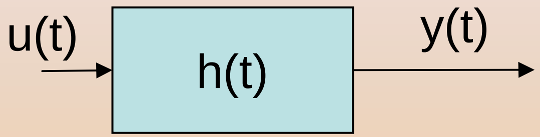
- Example : u(t) is an impulsion (0 everywhere except in t = 0)

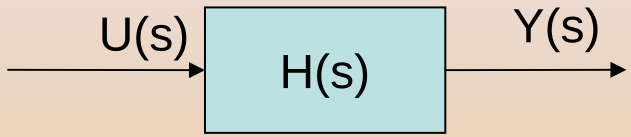
$$y(t) = h(t)$$

⇨ h(t) is called the **impulse response**, h(t) describes completely the system

- **Causality** : h(t) = 0 if t < 0

Laplace transform (1) : definition





Time space | **Laplace space**

t : real (time) ⟷ s : complex (frequency)

x(t) ⟶ $X(s) = \int_0^{+\infty} x(t) \cdot e^{-s \cdot t} \cdot dt$

$x(t) = \frac{1}{2 \cdot \pi \cdot j} \cdot \int_{c-j \cdot \infty}^{c-j \cdot \infty} X(s) \cdot e^{s \cdot t} \cdot ds$ ⟵ X(s)

$y(t) = \int_{-\infty}^{+\infty} u(\tau) \cdot h(t-\tau) \cdot d\tau$ ⟷ Y(s) = H(s) . U(s)

☺ Mathematical formulas are never used !

Laplace transform (2) : properties

Impulse fonction

t=0 : x(t) = infinite

x(t)=0

X(s) = 1

Step fonction :

t<0 : x(t) = 0

t>0 : x(t) = 1

X(s) = 1/s

Derivation :

$$y(t) = \frac{d}{dt} x(t)$$

$$Y(s) = s.X(s) - x(0_+)$$

Sinusoïdal fonction :

$$y(t) = \sin(\omega \cdot t)$$

$$Y(s) = \frac{1}{s^2 + \omega^2}$$

Laplace transform (3) : properties

Delay :

$$y(t) = x(t - t_d) \qquad\qquad Y(s) = X(s) \cdot e^{-t_d \cdot s}$$

Initial value theorem :

$$y(0_+) = \lim_{s \to \infty} (s \cdot Y(s))$$

Final value theorem (if limit exists) :

$$y(+\infty) = \lim_{s \to 0} (s \cdot Y(s))$$

Laplace transform (4) : tables

Table des transformées de Laplace

	f(t)	F(s)
P1	1 ou u(t)	$\frac{1}{s}$
P2	t	$\frac{1}{s^2}$
P3	t^n (n entier positif)	$\frac{n!}{s^{n+1}}$
P4	e^{-at}	$\frac{1}{s+a}$
P5	te^{-at}	$\frac{1}{(s+a)^2}$
P6	sin ωt	$\frac{\omega}{s^2+\omega^2}$
P7	cos ωt	$\frac{s}{s^2+\omega^2}$

From t to s

Laplace transform (4) : tables

	F(s)	f(t)
P27	$\frac{1}{(s-a)^n}$, n entier	$\frac{t^{n-1}e^{at}}{(n-1)!}$
P28	$\frac{1}{s^2+a^2}$	$\frac{\sin(at)}{a}$
P29	$\frac{1}{(s-b)^2+a^2}$	$\frac{e^{bt}\sin(at)}{a}$
P30	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\left(\frac{e^{at}-e^{-at}}{2}\right)$
P31	$\frac{s}{s^2-a^2}$	$\frac{e^{at}+e^{-at}}{2}$
P32	$\frac{1}{(s-a)(s-b)}$, si $a \neq b$	$\frac{e^{bt}-e^{at}}{b-a}$
P33	$\frac{s}{(s-a)(s-b)}$, si $a \neq b$	$\frac{be^{bt}-ae^{at}}{b-a}$
P34	$\frac{1}{(s^2+a^2)^2}$	$\frac{\sin(at)-at\cos(at)}{2a^3}$
P35	$\frac{s}{(s^2+a^2)^2}$	$\frac{t\sin(at)}{2a}$
P36	$\frac{s^2}{(s^2+a^2)^2}$	$\frac{\sin(at)+at\cos(at)}{2a}$

From s to t

Laplace transform and differential equations

$$a_o x(t) + a_1 \dot{x}(t) + a_2 \ddot{x}(t) = b_0 u(t) + b_1 \dot{u}(t)$$

+

Theorem of differentiation

⇓

$$a_o X(s) + a_1 \cdot \left(s \cdot X(s) - x(0_+)\right) + a_2 \left(s \cdot \left(s \cdot X(s) - x(0_+)\right) - \dot{x}(0_+)\right)$$
$$= b_0 U(s) + b_1 \left(s \cdot U(s) - u(0_+)\right)$$

Laplace transform and differential equations

$$a_o X(s) + a_1 \cdot (s \cdot X(s) - x(0_+)) + a_2 (s \cdot (s \cdot X(s) - x(0_+)) - \dot{x}(0_+))$$
$$= b_0 U(s) + b_1 (s \cdot U(s) - u(0_+))$$

⇓

$$\left(a_o + a_1 \cdot s + a_2 \cdot s^2\right) \cdot X(s) - (a_1 + a_2 \cdot s) \cdot x(0_+) - a_2 \cdot \dot{x}(0_+)$$
$$= (b_0 + b_1 \cdot s) \cdot U(s) - b_1 \cdot u(0_+)$$

⇓

$$X(s) = \frac{b_0 + b_1 \cdot s}{a_o + a_1 \cdot s + a_2 \cdot s^2} U(s) + \frac{(a_1 + a_2 \cdot s) \cdot x(0_+) + a_2 \cdot \dot{x}(0_+) - b_1 \cdot u(0_+)}{a_o + a_1 \cdot s + a_2 \cdot s^2}$$

→ Initial conditions

→ Transfer function

Laplace transform and differential equations

$$X(s)=\frac{b_0+b_1\cdot s}{a_o+a_1\cdot s+a_2\cdot s^2}U(s)+\frac{(a_1+a_2\cdot s)\cdot x(0_+)+a_2\cdot \dot{x}(0_+)-b_1\cdot u(0_+)}{a_o+a_1\cdot s+a_2\cdot s^2}$$

⇓

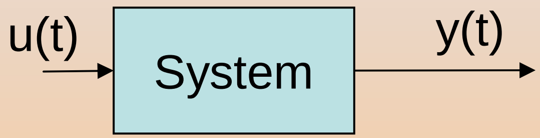
$$X(s)=H(s)\cdot U(s)+I(s)$$

⇕

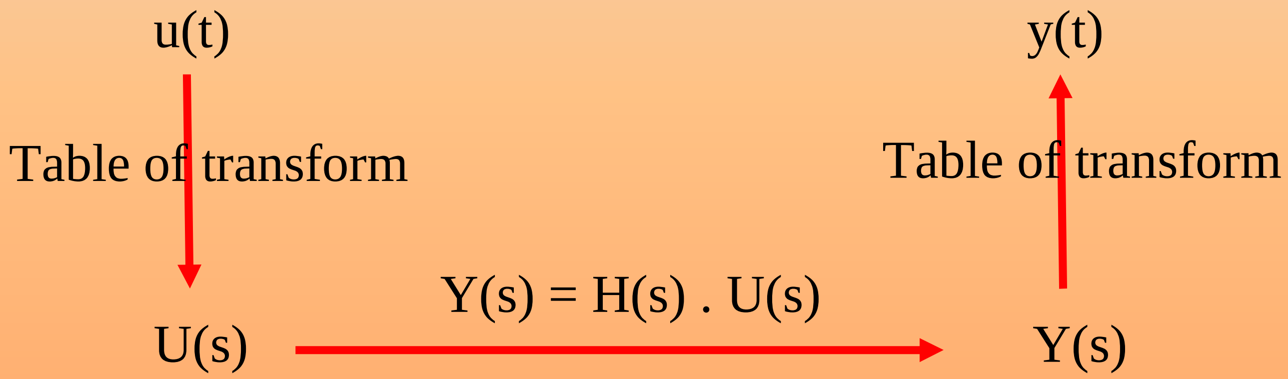
$$x(t)=h(t)*u(t)+i(t)$$

☺ Initial conditions are generally assumed to be null !

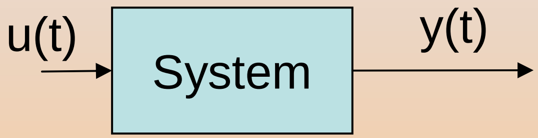
Finding output response with Laplace transform



What is the output y(t) from a given input u(t) ?



Finding final value with Laplace transform



What is the final output $y(\inf)$ from a given input $u(t)$?

$u(t)$

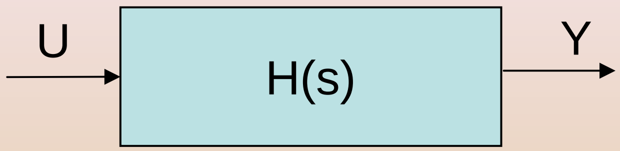
↓ Table of transform

$U(s)$ → $Y(s) = H(s) \,.\, U(s)$ → $Y(s)$

↑ Theorem of final value

$$y(+\infty) = \lim_{s \to 0}\left(s \cdot Y(s)\right)$$

Poles and zeros



Transfer function is a ratio of polynomials :

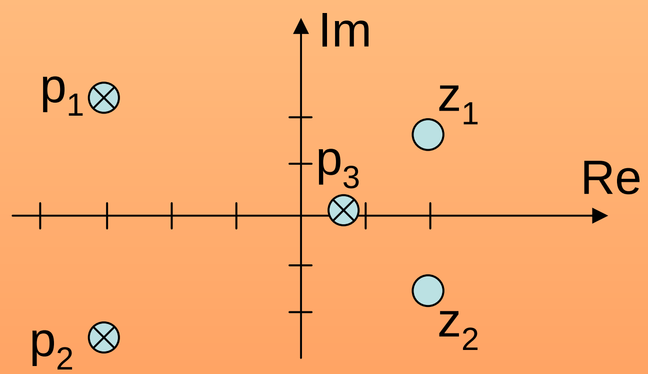
$$\frac{Y(s)}{U(s)} = H(s) = \frac{N(s)}{D(s)} = \frac{b_0 + b_1 s + b_2 s^2 + \ldots}{a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \ldots}$$

Poles and zeros :

$$\frac{Y(s)}{U(s)} = H(s) = \frac{b_0}{a_0} \cdot \frac{(s - z_1)\cdot(s - z_2)\cdots}{(s - p_1)\cdot(s - p_2)\cdot(s - p_3)\cdots}$$

zero : z_1, z_2, …

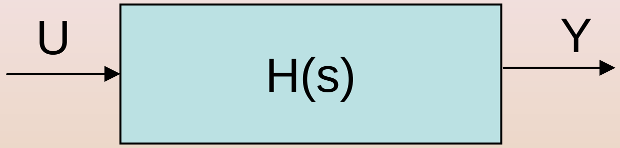
poles : p_1, p_2, p_3…



- Poles and zeros are either into the left plane ore into the right plane
- Complex poles and zeros have a conjugate

- Poles are the roots of Transfer function denominator
  - Real values or conjugate complex pairs
- Poles are also the eigenvalues of matrix A
- Poles = modes

Poles and zeros : decomposition



Transfer function can be expansed into a sum of elementary terms :

$$\frac{Y(s)}{U(s)} = H(s) = \frac{b_0 + b_1 s + b_2 s^2 + \ldots}{a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \ldots}$$

$$\frac{Y(s)}{U(s)} = H(s) = \frac{\alpha_1}{s - p_1} + \frac{\alpha_2}{s - p_2} + \frac{\alpha_3}{s - p_3} + \ldots$$

p_1 and p_2 are conjugate : $p_1 = -\omega_0 \cdot e^{j \cdot \theta}, p2 = -\omega_0 \cdot e^{-j \cdot \theta}$

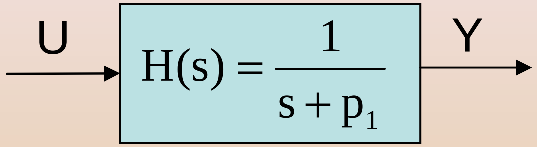
$$\frac{Y(s)}{U(s)} = H(s) = \frac{\alpha_{1,2}}{s^2 + 2 \cdot \omega_0 \cdot \cos\theta \cdot s + \omega_0^2} + \frac{\alpha_3}{s - p_3} + \ldots$$

First orders

Second orders

☺ Complex system response is the sum of first order and second order systems responses

Dynamic response of first order systems



$$Y(s) = \frac{1}{s + p_1} U(s)$$

Example 1 : impulse response

u(t) is an impulsion (0 everywhere, except in 0 : ∞)

$u(t)$

Table of transform

$U(s)=1$

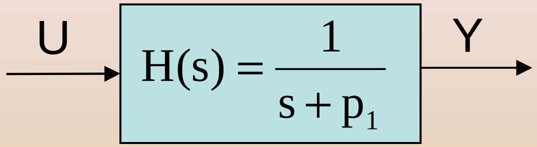
$$Y(s) = H(s) \cdot U(s)$$

$$Y(s) = \frac{1}{s + p_1}$$

Table of transform

$y(t) = e^{-p_1 \cdot t}$

Dynamic response of first order systems



$$Y(s) = \frac{1}{s + p_1} U(s)$$

Example 2 : step response

u(t) is a step

$$u(t)$$

Table of transform

$$U(s) = \frac{1}{s}$$

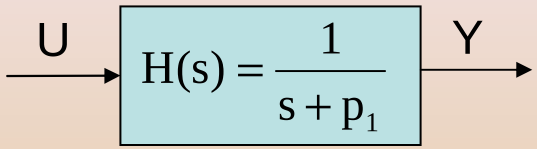
$$Y(s) = H(s) \cdot U(s)$$

$$Y(s) = \frac{1}{s + p_1} \cdot \frac{1}{s}$$

Table of transform

$$y(t) = \frac{1}{p_1}\left(1 - e^{-p_1 \cdot t}\right)$$

Properties of first order systems

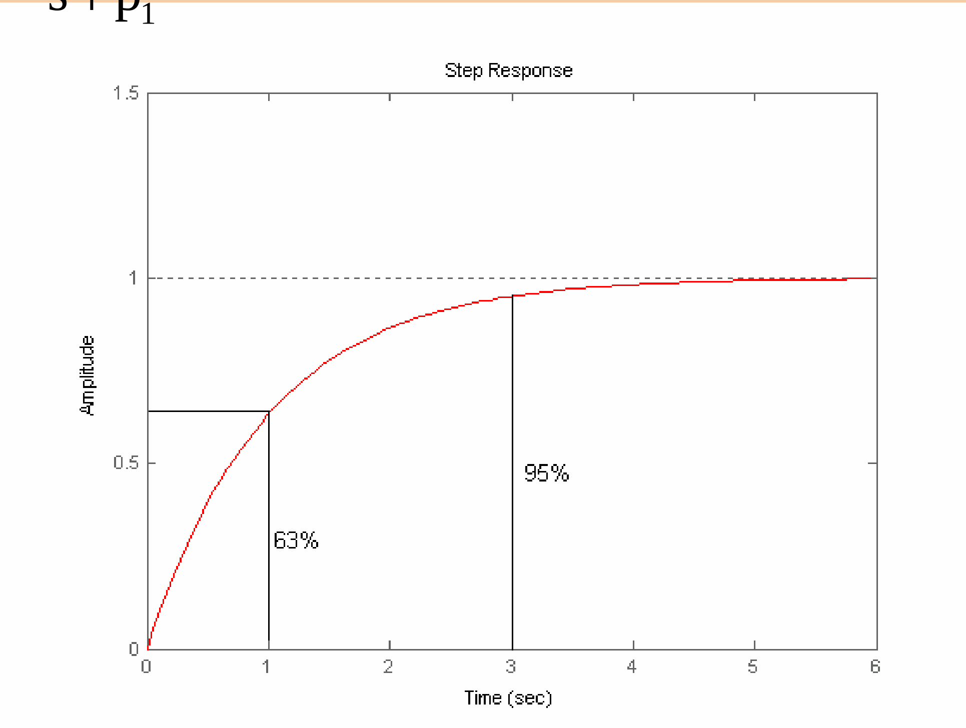


$$Y(s) = \frac{1}{s+p_1} U(s)$$

Step response

$t_1 = 1/p_1$ is the **time constant** of the system :

after $t = t_1$, 63% of the final value is obtained



Dynamic response of second order systems

U → $H(s) = \frac{1}{s^2 + 2 \cdot \sigma \cdot \omega_0 \cdot s + \omega_0^2}$ → Y

Example 1 : impulse response

u(t) is an impulsion (0 everywhere, except in 0 : ∞)

$u(t)$

Table of transform

$U(s)=1$

$$Y(s) = H(s) \cdot U(s)$$

$$Y(s) = \frac{1}{s^2 + 2 \cdot \sigma \cdot \omega_0 \cdot s + \omega_0^2}$$

Table of transform

$$y(t) = \frac{1}{\omega_0 \sqrt{1-\sigma^2}} \cdot e^{-\sigma \cdot \omega_0 \cdot t} \cdot \sin\left(\omega_0 \sqrt{1-\sigma^2} \cdot t\right)$$

Dynamic response of second order systems

U → $H(s) = \dfrac{1}{s^2 + 2 \cdot \sigma \cdot \omega_0 \cdot s + \omega_0{}^2}$ → Y

Example 2 : step response
u(t) is an step

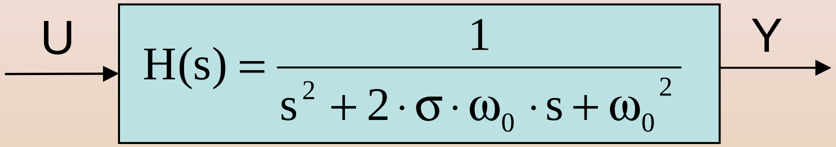
$$u(t) \qquad y(t) = 1 - \frac{1}{\omega_0 \sqrt{1-\sigma^2}} \cdot e^{-\sigma \cdot \omega_{\text{à}} \cdot t} \cdot \sin\left(\omega_0 \sqrt{1-\sigma^2} \cdot t + \arccos(\sigma)\right)$$

Table of transform ↓ Table of transform ↑

$$Y(s) = H(s) \cdot U(s)$$

$$U(s) = 1/s \quad \longrightarrow \quad Y(s) = \frac{1}{s^2 + 2 \cdot \sigma \cdot \omega_0 \cdot s + \omega_0{}^2} \cdot \frac{1}{s}$$

Properties of second order systems



$$H(s) = \frac{1}{s^2 + 2\cdot\sigma\cdot\omega_0\cdot s + \omega_0^2}$$

Step response : $y(t) = 1 - \frac{1}{\omega_0\sqrt{1-\sigma^2}}\cdot e^{-\sigma\cdot\omega_0\cdot t}\cdot\sin\left(\omega_0\sqrt{1-\sigma^2}\cdot t + \arccos(\sigma)\right)$

σ is the **damping factor**

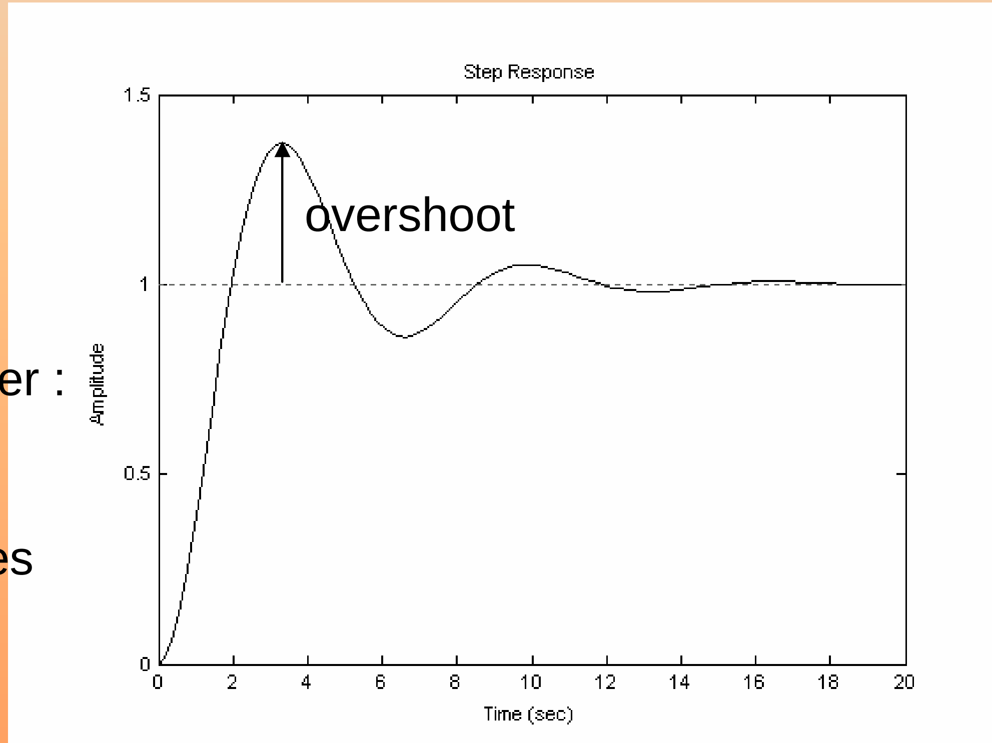
ω_0 is the **natural frequency**

$\omega_0\sqrt{1-\sigma^2}$ is the pseudo-frequency

5% of the final value is obtained after :

$$t_{5\%} \approx \frac{3}{\sigma\cdot\omega_0}$$

Overshoot increases as σ decreases

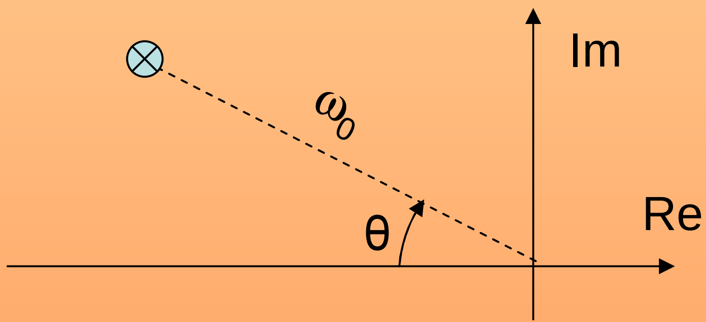


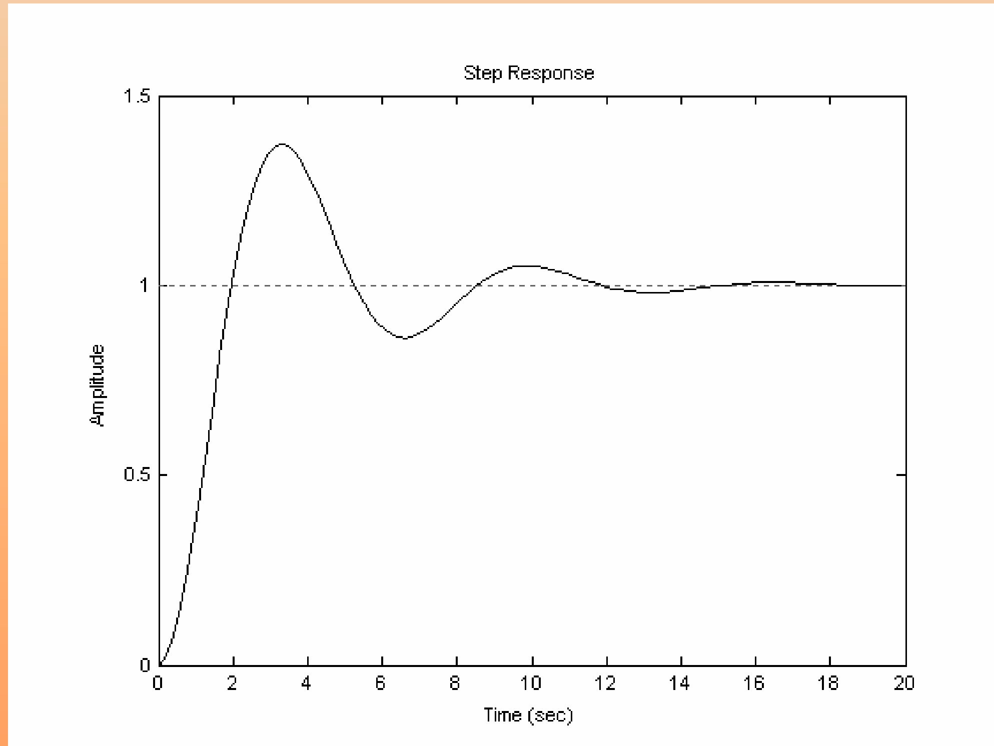
Properties of second order systems

U → $H(s) = \dfrac{1}{s^2 + 2 \cdot \sigma \cdot \omega_0 \cdot s + \omega_0^2}$ → Y

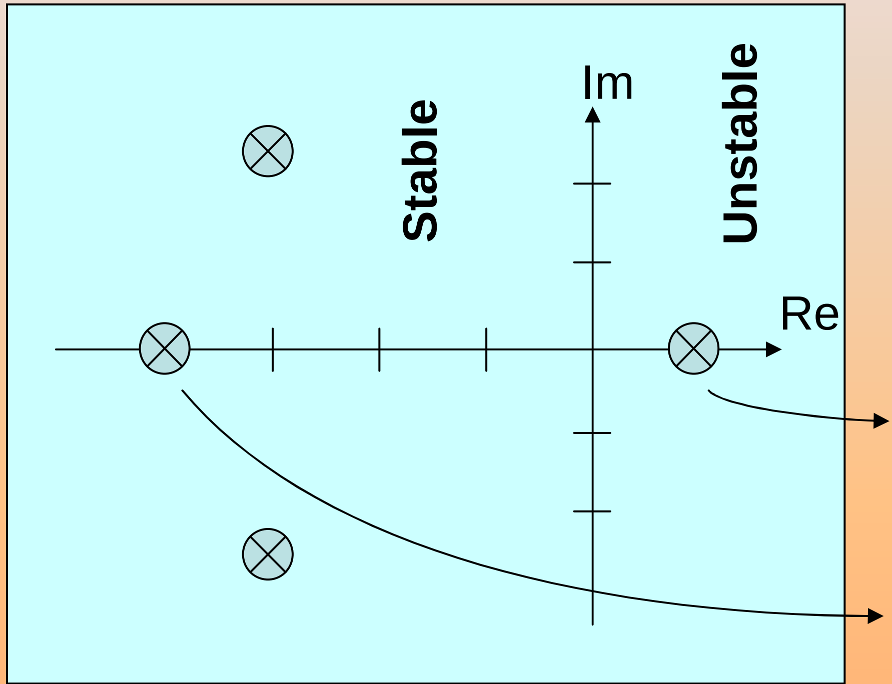
Step response (continued) :

poles : $p_{1,2} = -\omega_0 \cdot e^{\pm\theta}$ cos(θ)=σ





Stability



unstable pole, deverges like exp(t)

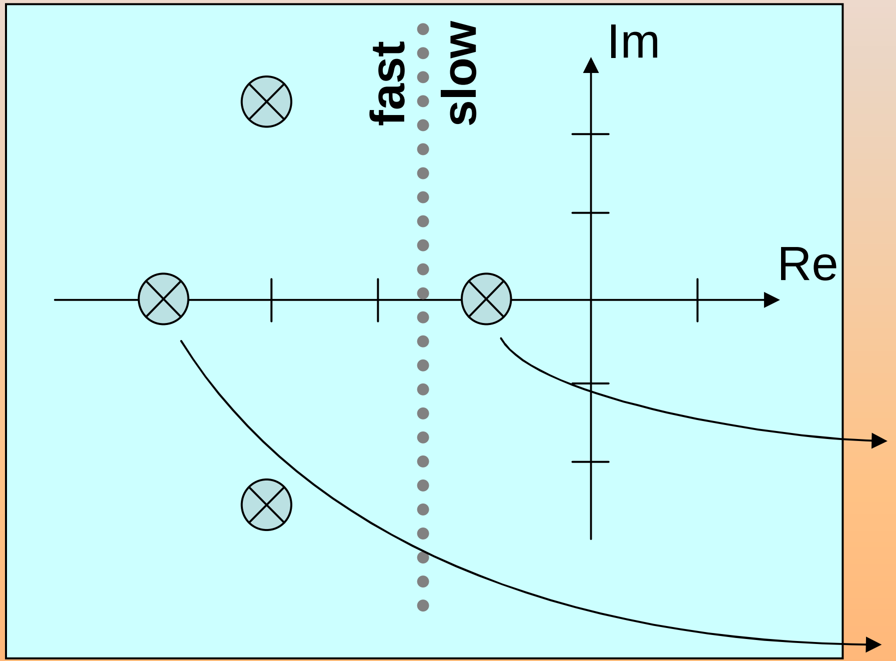
stable pole, decays like exp(-4.t)

Any pole with positive real part is unstable

⇨ **Any input (even small) will lead to instability**

[See animation](#)

« fast poles » vs « slow poles »



slow pole, decays like
exp(-t)
constant time : t_1 = 1s

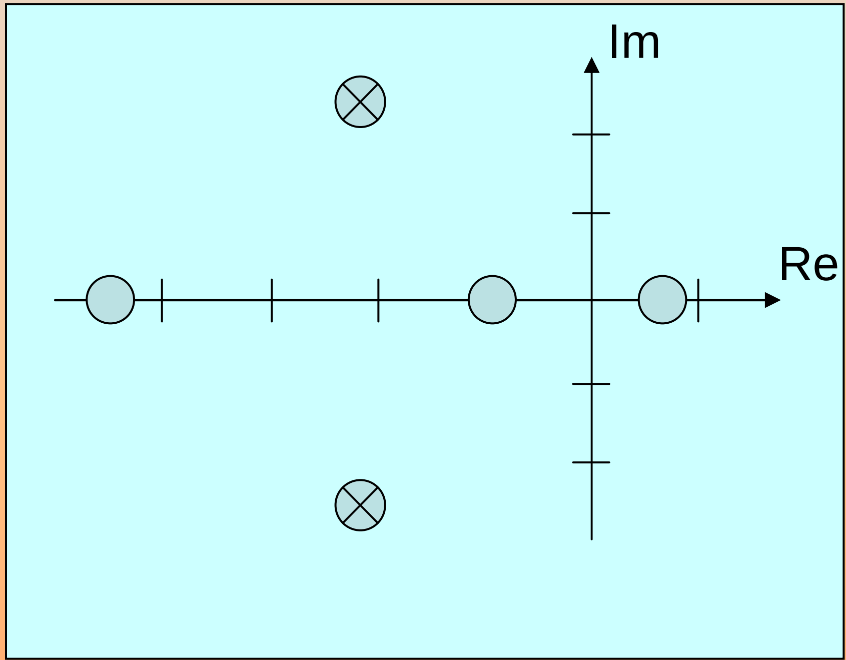
fast pole, decays like
exp(-4.t)
constant time : t_1 = 4s

Fast poles can be neglected

See animation

Effect of zeros

See animation



- Fast zero : neglected
- Slow zero : transient response affected
- Positive zero : non minimal phase system, step response start out in the wrong direction

Zeros modify the transient response

Ex. analysis of a feedback system

Process model :

$$\frac{dv(t)}{dt} + 0.02 \cdot v = u - 10 \cdot \theta$$

Transfer function**s** :

$$\begin{cases} s \cdot V(s) + 0.02 \cdot V(s) = U(s) \\ s \cdot V(s) + 0.02 \cdot V(s) = -10 \cdot \theta(s) \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{V(s)}{U(s)} = F(s) = \dfrac{1}{0.02 + s} \\ \dfrac{V(s)}{\theta(s)} = -\dfrac{10}{0.02 + s} \end{cases}$$

Ex. analysis of a feedback system

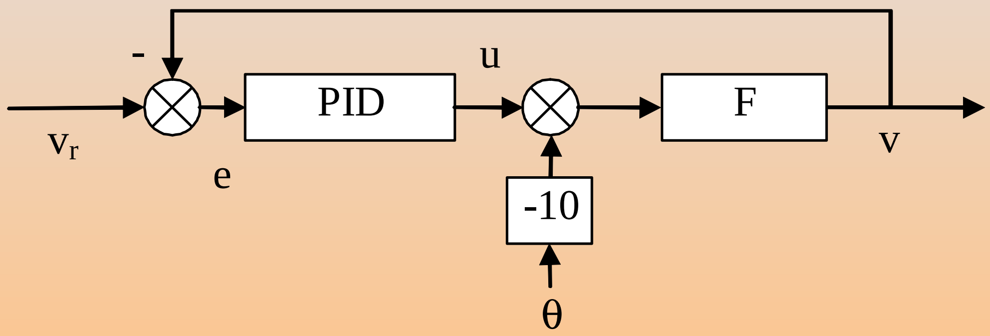
Transfer function of the controller (PID) :

$$u(t) = k \cdot e(t) + k_d \cdot \frac{de(t)}{dt} + k_i \cdot \int_0^t e(t) \cdot d\tau$$

$$\Rightarrow \frac{U(t)}{E(t)} = k + k_d \cdot s + k_i \cdot \frac{1}{s}$$

Ex. analysis of a feedback system

We can now combine transfer functions :



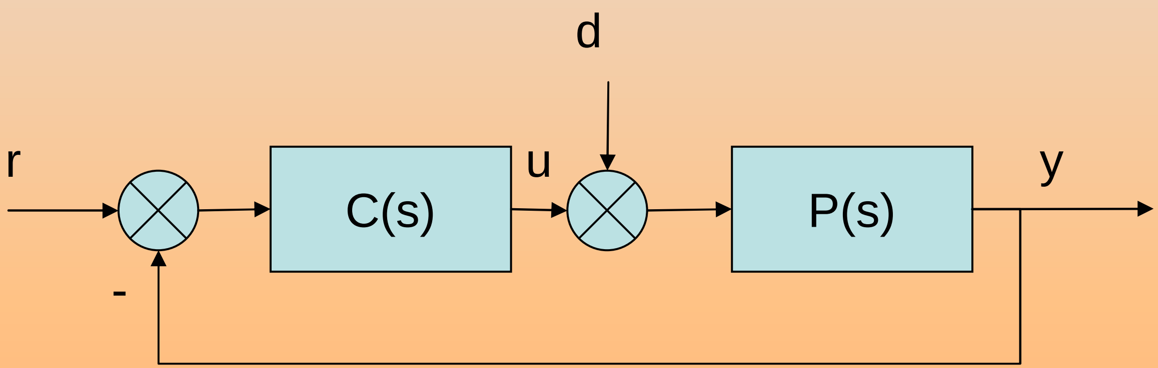
$$V(s) = \frac{1}{1 + F(s)\cdot PID(s)} \cdot V_r(s) + \frac{-10}{1 + F(s)\cdot PID(s)} \cdot E(s)$$

IV. Design of simple feedback

Introduction

Standard problems are often first orders or second orders

- Standard problem → standard solution



$$P(s) = \frac{b}{s+a} \qquad P(s) = \frac{b_1 s + b_2}{s^2 + a_1 \cdot s + a_2}$$

Control of a first order system

Most physical problems can be modeled as first order systems

Step 1 : transform your problem in a first order problem :

$$P(s) = \frac{b}{s+a}$$

Step 2 : choose a PI controller

$$C(s) = k + \frac{k_i}{s}$$

Step 3 : combine equations and tune k and in k_i in order to achieve the desired closed loop behavior (mass-spring damper analogy)

$$CL(s) = \frac{P(s)\cdot C(s)}{1+P(s)\cdot C(s)} = \frac{\frac{b}{s+a}\cdot\left(k+\frac{k_i}{s}\right)}{1+\frac{b}{s+a}\cdot\left(k+\frac{k_i}{s}\right)} = K\cdot\frac{1+b'\cdot s}{1+\frac{2\cdot\sigma}{\omega_0}\cdot s+\frac{s^2}{{\omega_0}^2}}$$

Dynamic response of second order systems

$$U \rightarrow \boxed{H(s) = \frac{1}{s^2 + 2 \cdot \sigma \cdot \omega_0 \cdot s + \omega_0{}^2}} \rightarrow Y$$

Example 1 : impulse response

u(t) is an impulsion (0 everywhere, except in 0 : ∞)

$u(t)$

↓ Table of transform

$U(s)=1$

$$Y(s) = H(s) \cdot U(s)$$

→ $Y(s) = \dfrac{1}{s^2 + 2 \cdot \sigma \cdot \omega_0 \cdot s + \omega_0{}^2}$

↑ Table of transform

$$y(t) = \frac{1}{\omega_0 \sqrt{1-\sigma^2}} \cdot e^{-\sigma \cdot \omega_{\grave{a}} \cdot t} \cdot \sin\left(\omega_0 \sqrt{1-\sigma^2} \cdot t\right)$$

Dynamic response of second order systems

U → $H(s) = \frac{1}{s^2 + 2 \cdot \sigma \cdot \omega_0 \cdot s + \omega_0{}^2}$ → Y

Example 2 : step response
u(t) is an step

$$u(t)$$

$$y(t) = 1 - \frac{1}{\omega_0 \sqrt{1-\sigma^2}} \cdot e^{-\sigma \cdot \omega_{à} \cdot t} \cdot \sin\left(\omega_0 \sqrt{1-\sigma^2} \cdot t + \mathrm{ar}\cos(\sigma)\right)$$

Table of transform

Table of transform

$$Y(s) = H(s) \cdot U(s)$$

U(s)=1/s

$$Y(s) = \frac{1}{s^2 + 2 \cdot \sigma \cdot \omega_0 \cdot s + \omega_0{}^2} \cdot \frac{1}{s}$$

Properties of second order systems

$$U \rightarrow \boxed{H(s) = \frac{1}{s^2 + 2\cdot\sigma\cdot\omega_0\cdot s + \omega_0^2}} \rightarrow Y$$

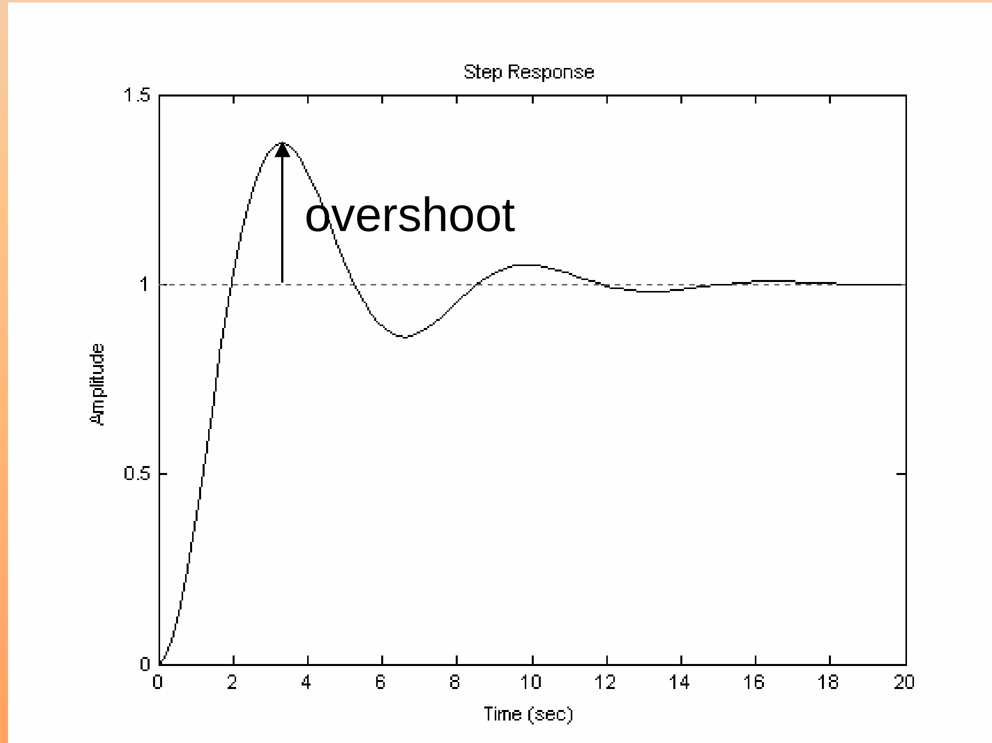
Step response : $y(t) = 1 - \frac{1}{\omega_0\sqrt{1-\sigma^2}}\cdot e^{-\sigma\cdot\omega_0\cdot t}\cdot\sin\left(\omega_0\sqrt{1-\sigma^2}\cdot t + \arccos(\sigma)\right)$

σ is the **damping factor**
ω_0 is the **natural frequency**
$\omega_0\sqrt{1-\sigma^2}$ is the pseudo-frequency

5% of the final value is obtained after :

$$t_{5\%} \approx \frac{3}{\sigma\cdot\omega_0}$$

Overshoot increases as σ decreases

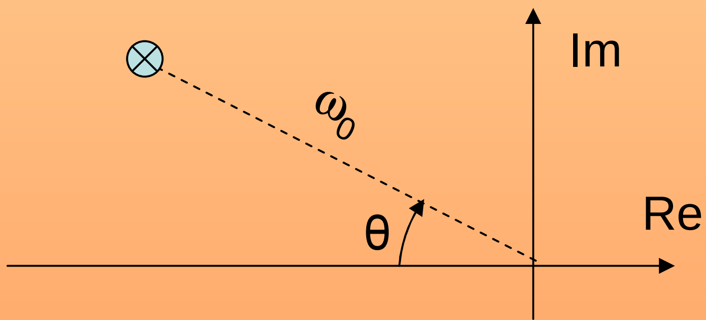


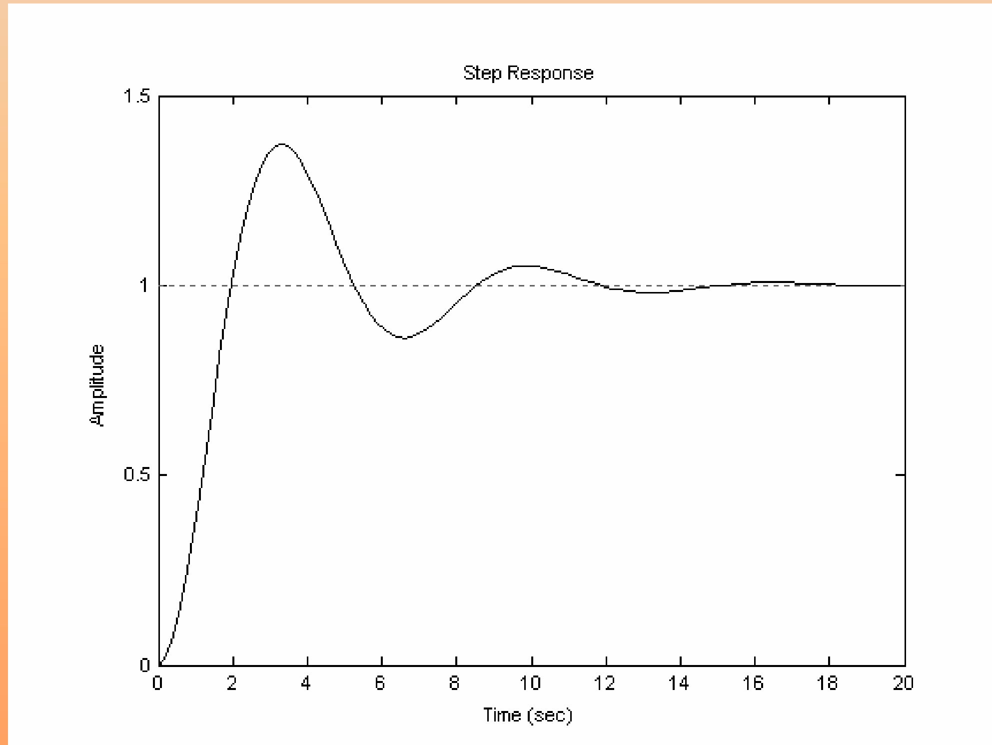
Properties of second order systems

U → $H(s) = \dfrac{1}{s^2 + 2 \cdot \sigma \cdot \omega_0 \cdot s + \omega_0^2}$ → Y

Step response (continued) :

poles : $p_{1,2} = -\omega_0 \cdot e^{\pm\theta}$ $\cos(\theta)=\sigma$





Control of a second order system

Step 1 , step 2 : idem (PI controller)

Step 3 : Transfer function is now third order ☹

$$CL(s) = \frac{P(s)\cdot C(s)}{1+P(s)\cdot C(s)} = K \cdot \frac{1+b'\cdot s}{(1+a\cdot s)\left(1+\frac{2\cdot\sigma}{\omega_0}\cdot s+\frac{s^2}{\omega_0^{\ 2}}\right)}$$

2 dof (k and k_i) : the full dynamics (order 3) cannot be totally chosen →

Simulation tools

Matlab or Scilab

→ Transfer function is a Matlab object

→ Adapted to transfer function algebra (addition, multiplication…)

→ Simulation, time domain analysis

Conclusion

Laplace Transform

+

Simulation tools

➔ Design of simple feedbacks

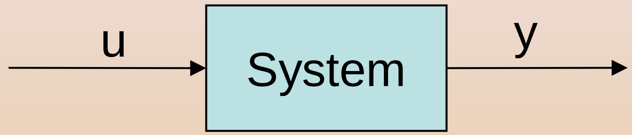
V. Frequency response

Introduction

Frequency response :

- One way to view dynamics
- Heritage of electrical engineering (Bode)
- Fits well block diagrams
- Deals with systems having large order
  - electronic feedback amplifier have order 50-100 !
- input output dynamics, black box models, external description
- Adapted to experimental determination of dynamics

The idea of black box



The system is a black box : forget about the internal details and focus only on the input-output behavior

→ Frequency response makes a “giant table” of possible inputs-outputs pairs

→ Test entries are enough to fully describe LTI systems ☺

- Step response
- Impulse response
- sinusoids

What is a LTI system

A Linear Time Invariant System is :

- Linear

  → If (u_1,y_1) and (u_2,y_2) are input-output pairs then $(a.u_1+ b.u_2\ ,\ a.y_1+ b.y_2)$ is an input-output pair : Theorem of superposition

- Time Invariant

  → $(u_1(t),y_1(t))$ is an input-output pair then $(u_1(t-T),y_1(t-T))$ is an input-output pair

The "giant table" is drastically simplified :

$$y(t) = \int_{-\infty}^{+\infty} h(t-\tau)\cdot u(\tau)\cdot d\tau$$

$$\Rightarrow Y(s) = H(s)\cdot U(s)$$

What is the Fourier Transform

Fourier's idea : an LTI system is completely determined by its response to sinusoidal signals

- Transmission of sinusoid is given by G(jω)
- The transfer function G(s) is uniquely given by its values on the imaginary axes
- Frequency response can be experimentally determined

The complex number G(jω) tells how a sinusoid propagates through the system *in steady states* :

$$u(t) = \sin(\omega \cdot t)$$

$$\Rightarrow y(t) = |G(j \cdot \omega)| \cdot \sin(\omega \cdot t + \arg(G(j \cdot \omega)))$$

Steady state response

Fourier transform deals with Steady State Response :

$$u(t) = \cos(\omega_0 \cdot t) + i \cdot \sin(\omega_0 \cdot t) = e^{i \cdot \omega_0 \cdot t}$$

$$\Rightarrow U(s) = \frac{1}{s - i \cdot \omega_0}$$

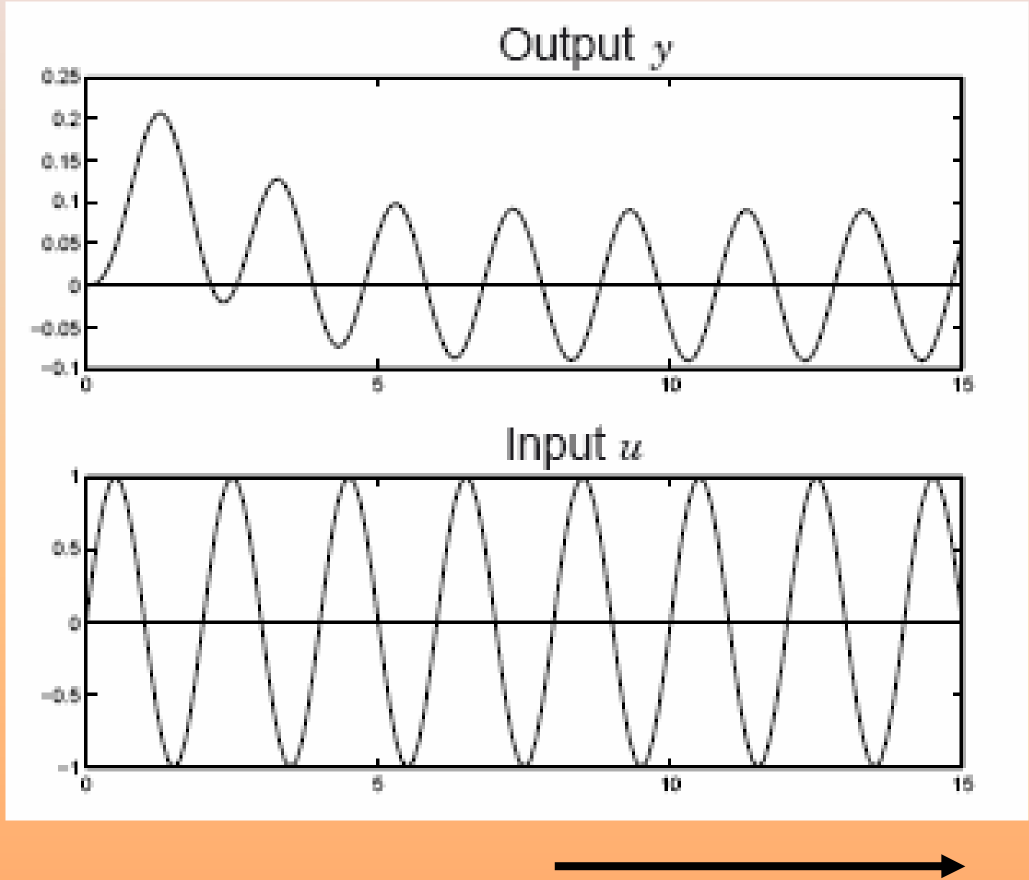
$$\Rightarrow Y(s) = G(s) \cdot \frac{1}{s - i \cdot \omega_0} = \frac{G(i \cdot \omega_0)}{s - i \cdot \omega_0} + \sum \frac{R_k}{s - \alpha_k}$$

(System has distinct poles α_k)

$$\Rightarrow y(t) = G(i \cdot \omega_0) \cdot e^{i \cdot \omega_0 \cdot t} + \sum R_k \cdot e^{\alpha_k \cdot t}$$

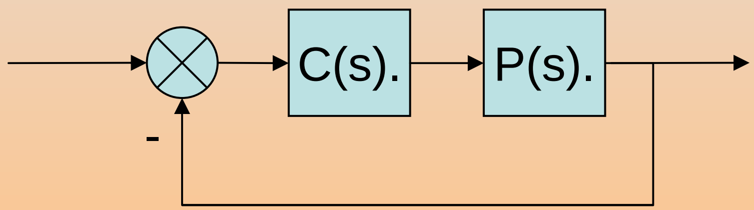
Decays if all α_k are negatives

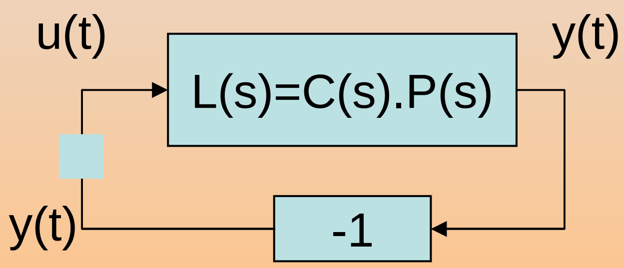
Steady state response



Nyquist stability theorem

Nyquist stability theorem tells if a system WILL BE stable (or not) with a simple feedback





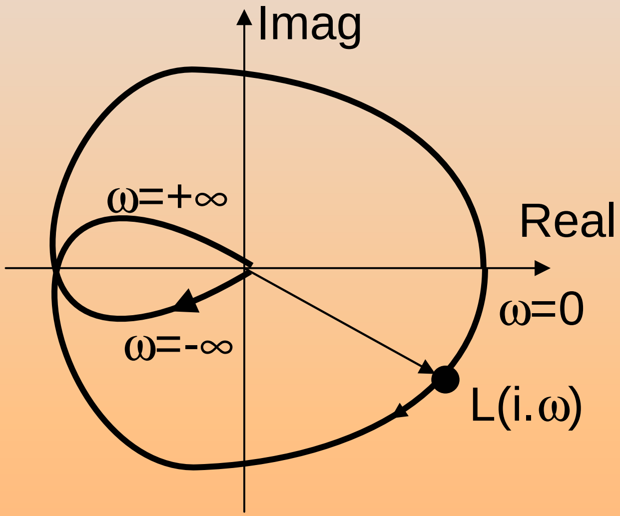
(1) Standard system with negative unitary feedback

(2) Nyquist standard form

(2) : if $L(i.\omega_0) = -1$ then oscillation will be maintained

Nyquist stability theorem

Step 1 : draw Nyquist curve



Step 2 : where is (-1,0) ?

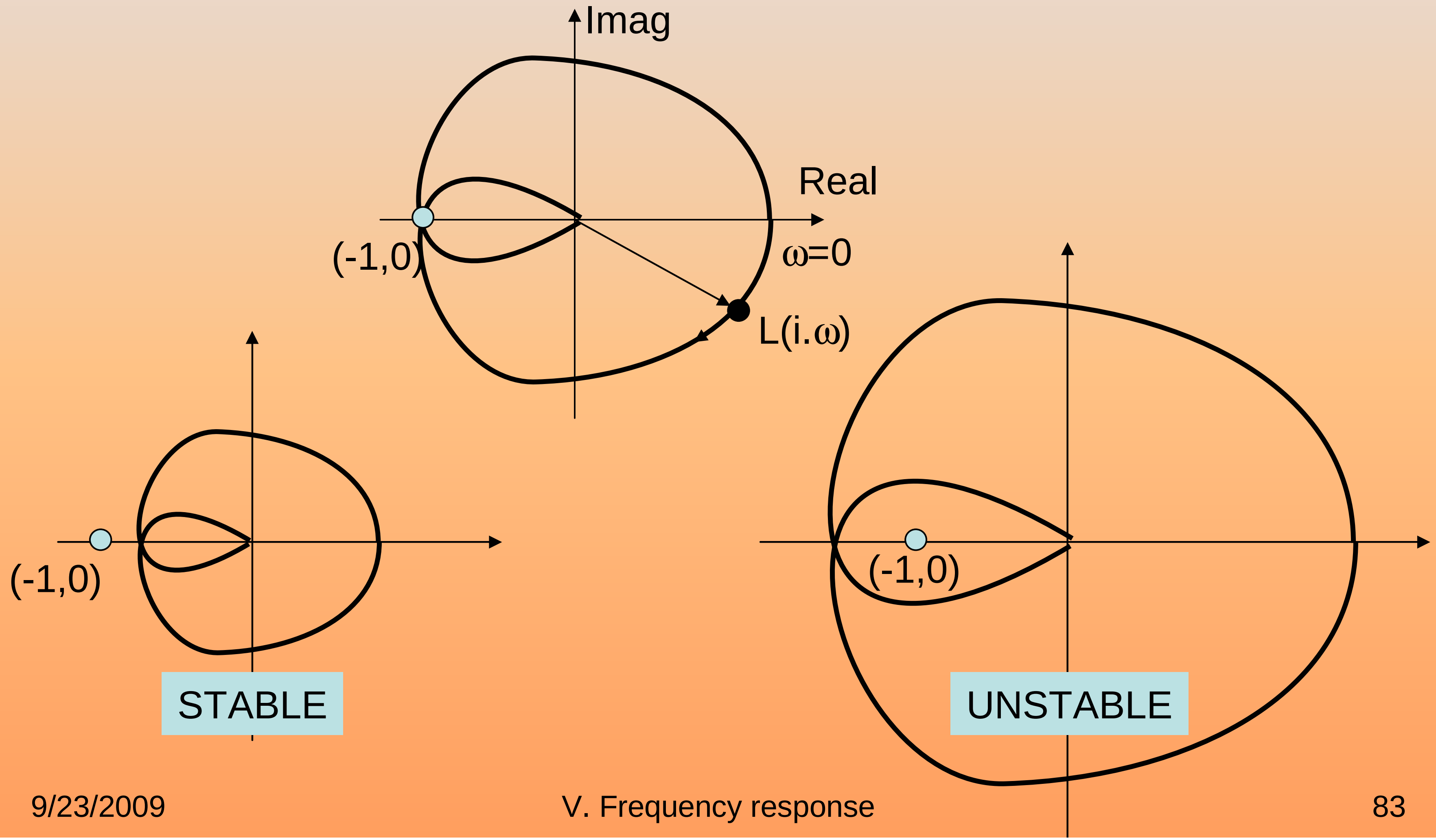
Nyquist theorem

When the transfer loop function L does not have poles in the right half plane *the closed loop system is stable if the complete Nyquist curve does not encircle the critical (-1,0) point.*

When the transfer loop function L has N poles in the right half plane *the closed loop system is stable if the complete Nyquist curve encircle the critical (-1,0) point N times.*

Nyquist stability theorem

Nyquist stability theorem compares L(i.ω) with (-1,0)

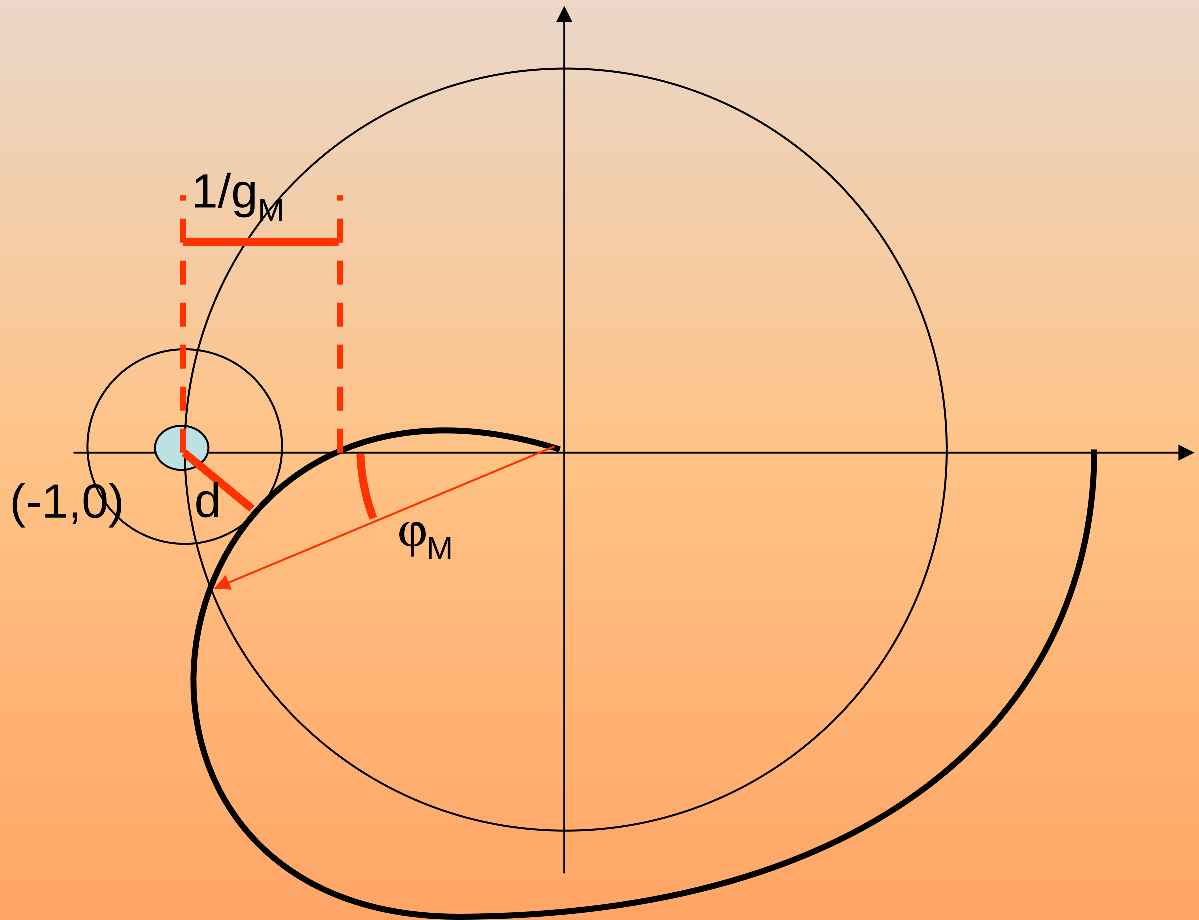


Nyquist theorem

- Focus on the characteristic equation
- Difficult to see how the characteristic equation L is influenced by the controller C
  - → Question is : how to change C ?
- Strong practical applications
- Possibility to introduce stability margin : how close to instability are we ?

Stability margin

Stability margin definitions :



φ_M Phase margin

45°- 60°

g_M Gain margin

2 - 6

d Shortest distance to critical point

0.5 - 0.8

The Bode plot

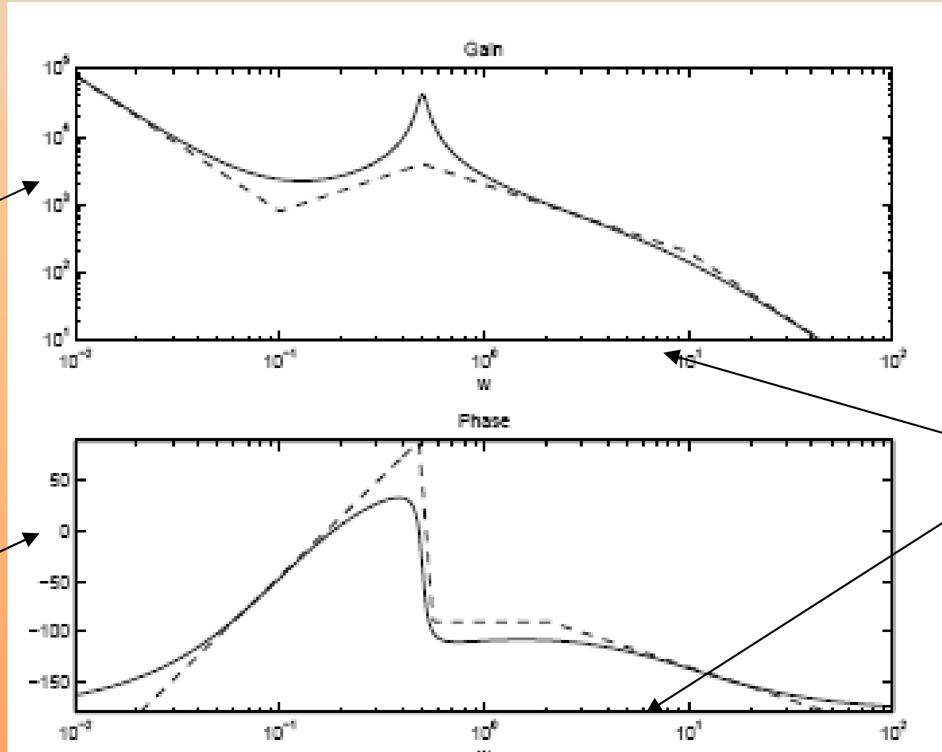
Nyquist theorem is spectacular but not very efficient…

→ Impossible to distinguish C(s) and P(s)

Bode plots two curves : one for gain, one for phase :

dB scale for gain

Linear scale for phase



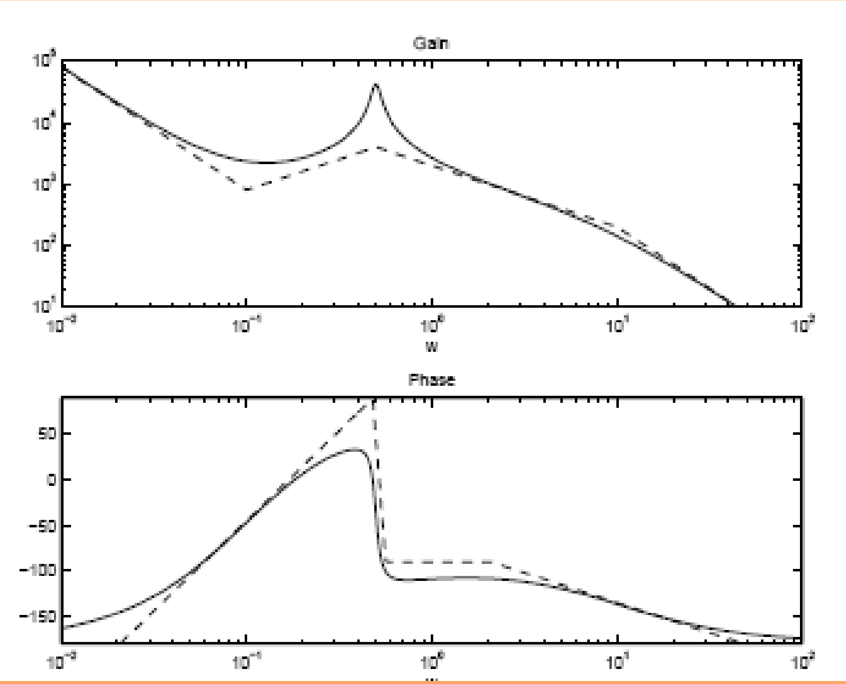
Logarithmic frequency axis

The Bode plot

Bode's plot main properties :

→ Asymptotic curves (gain multiple of 20dB/dec) are ok

→ Simple interpretation of C(s) and P(s) in cascade :



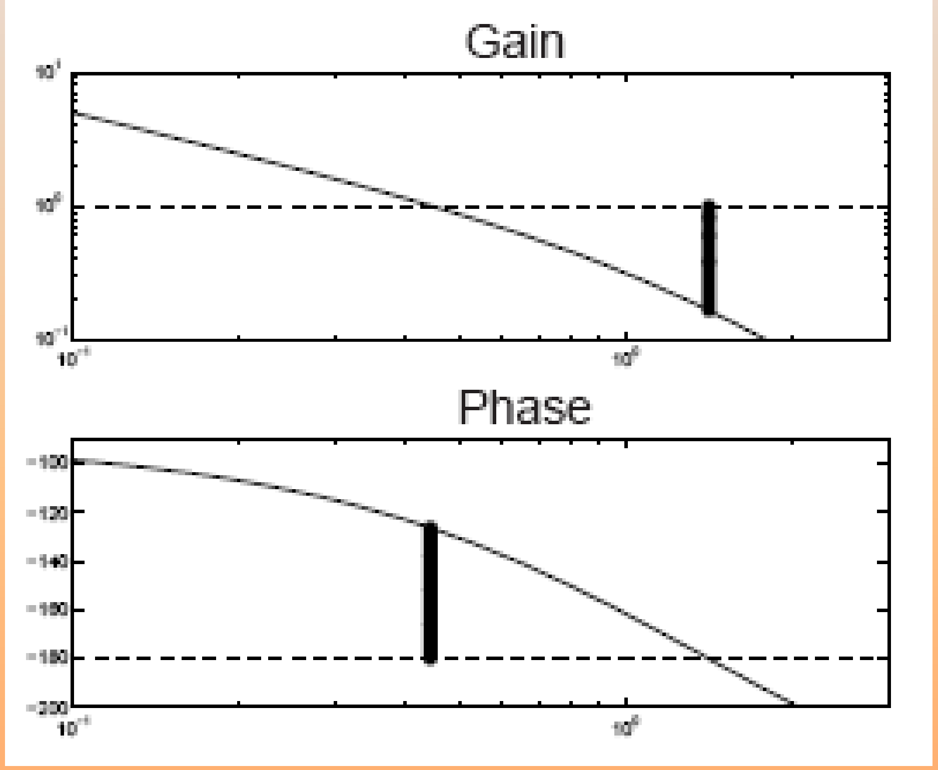
$Gain_{dB}(C(s).P(s)$

$= Gain_{dB}(C(s)) + Gain_{dB}(P(s))$

Phase(C(s).P(s)

= Phase(C(s)) + Phase(P(s))

The Bode stability criteria



Gain Margin > 0 : closed loop system will be stable

Phase Margin > 0 : closed loop system will be stable

One criteria is sufficient in most cases because gain and margin are closely related

Close loop frequency response

Typical property of $H_{bo}(s)$ are :

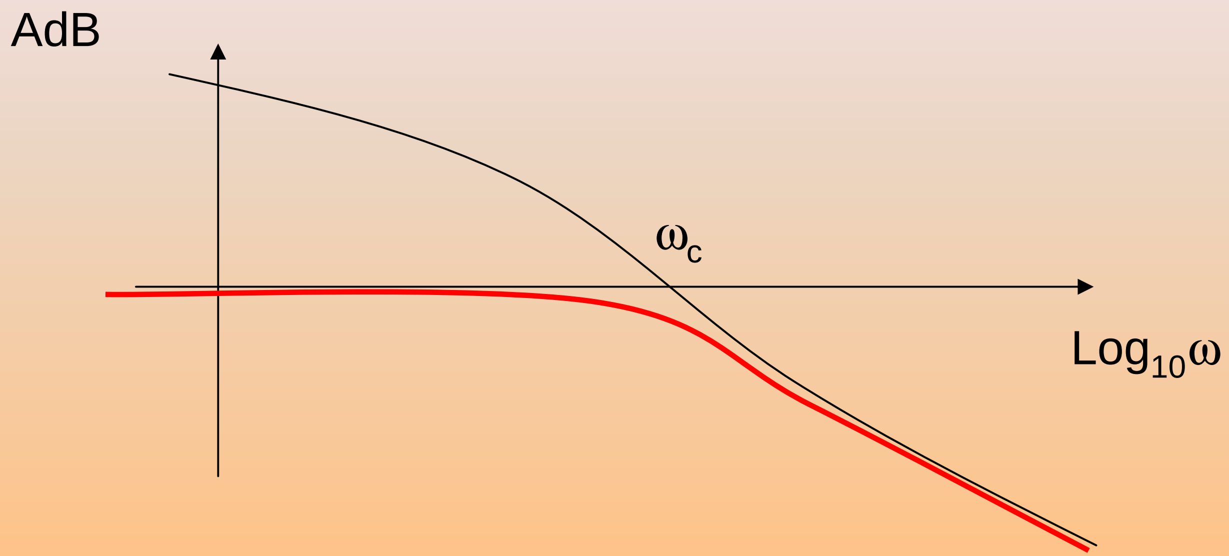
$$\left|H_{bo}(j\cdot\omega)\right| >> 1 \text{ for } \omega << \omega_c$$

$$\left|H_{bo}(j\cdot\omega)\right| << 1 \text{ for } \omega >> \omega_c$$

$$H_{bf}(j\cdot\omega) = \frac{H_{bo}(j\cdot\omega)}{1+H_{bo}(j\cdot\omega)} \approx 1 \text{ for } \omega << \omega_c$$

$$H_{bf}(j\cdot\omega) = \frac{H_{bo}(j\cdot\omega)}{1+H_{bo}(j\cdot\omega)} \approx H_{bo}(j\cdot\omega) \text{ for } \omega >> \omega_c$$

Close loop frequency response

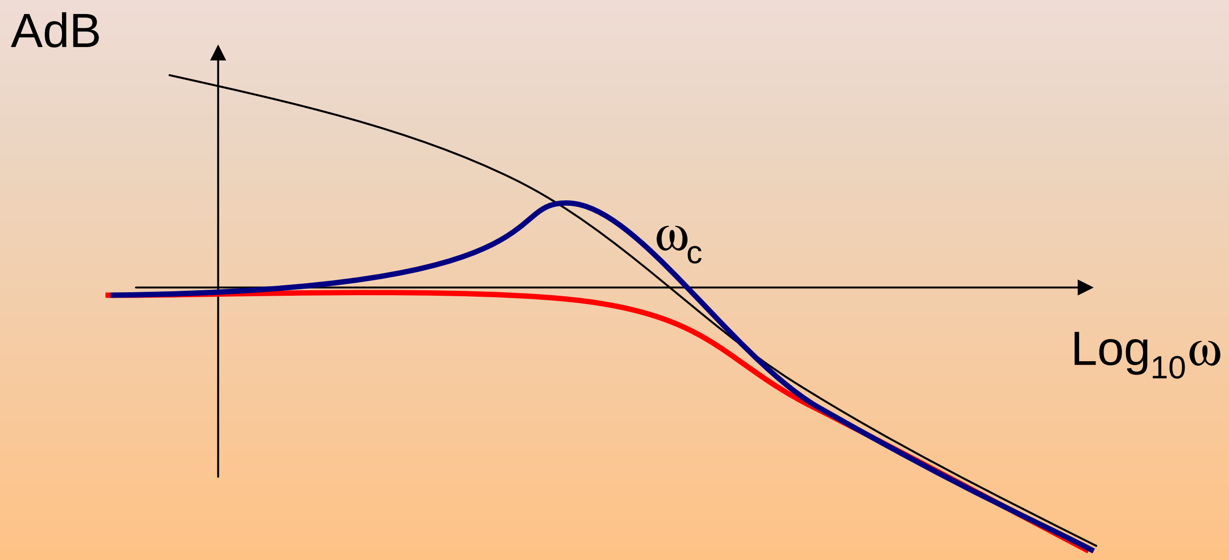


— Open loop Bode plot

— Close loop Bode plot

Close loop frequency response

Phase margin effect



Open loop Bode plot

Close loop Bode plot (phase margin 90°)

Close loop Bode plot (phase margin 30°)

Static gain

$$H_{bf}(j\cdot\omega)=\frac{H_{bo}(j\cdot\omega)}{1+H_{bo}(j\cdot\omega)}$$

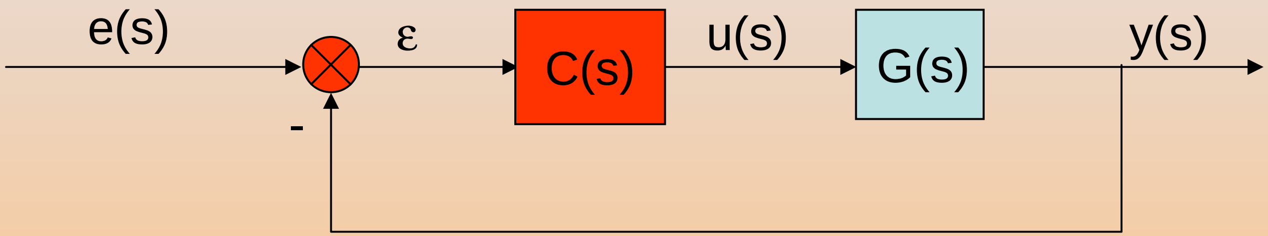
For small value of ω (low frequency) :

If $H_{bo}(\omega) << 1$ then $H_{bf}(\omega) \cong H_{bo}(\omega)$

If $H_{bo}(\omega) >> 1$ then $H_{bf}(\omega) \cong 1$

If $H_{bo}(\omega) \to \infty$ then $H_{bf}(\omega) \to 1$

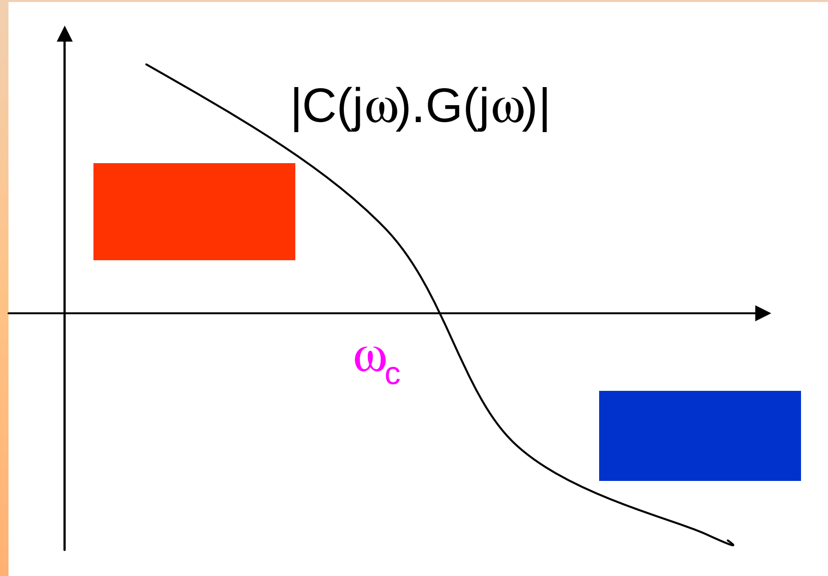
Controller specifications



Open loop transfer function : OLTF = C.G
Close loop transfer function : CLTF = C.G / (1 + C.G)

Controller specifications

- Static gain close to 1
  - → Low frequency : high gain
- Perturbation rejection
  - → High frequency : low gain
- Stability
  - → phase margin > 0
- Bandwith
  - → Cross over frequency ω_c
- Overshoot ≈ 25%
  - → phase margin ≈ 45°
  - → Gentle slope in transition region



Controller design

- Proportionnal feedback $C(s) = K$
  - Effect : lifts gain with no change in phase
  - Bode : shift gain by factor of K

Controller design

- Lead compensation

$$C(s) = K\frac{1+\tau \cdot s}{1+A \cdot \tau \cdot s}$$

  - Effect : lifts phase by increasing gain at high frequency
  - Very usefull controller : increase phase margin
  - Bode : add phase between zero and pole

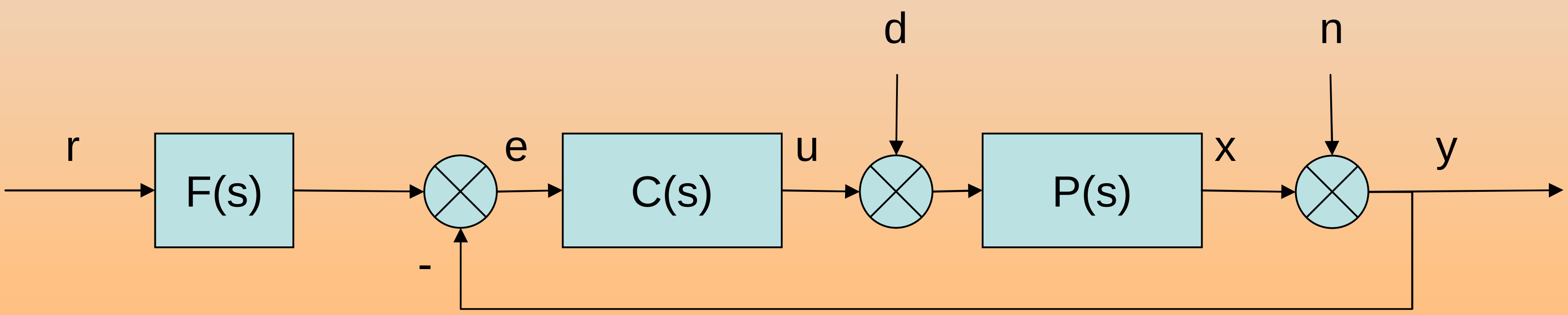
Modern loop shaping

- Use of rltool (Matlab Control Toolboxe)

VI. Design of simple feedback (Ctd)

Introduction

More complete standard problem :



→ Controller : feedback C(s) and feedforward F(s)

→ Load disturbance d : drives the system from its desired state x

→ Measurement disturbance n : corrupts information about x

→ Main requirement is that process variable x should follow reference r

Introduction

Controller's specifications :

A. Reduce effects of load disturbance
B. Does not inject too much measurement noise into the system
C. Makes the closed loop insensitive to variations in the process
D. Makes output follow reference signal

Classical approach : deal with A,B and C with controller C(s) and deal with D with feedforward F(s)

Introduction

Controller's specifications :

A. Reduce effects of load disturbance
B. Does not inject too much measurement noise into the system
C. Makes the closed loop insensitive to variations in the process
D. Makes output follow reference signal

Classical approach : deal with A,B and C with controller C(s) and deal with D with feedforward F(s) :

Design procedure

- Design the feedback C(s) too achieve
  - Small sensitivity to load disturbance d
  - Low injection of measurement noise n
  - High robustness to process variations
- Then design F(s) to achieve desired response to reference signal r

Relations between signals

Three interesting signals (x, y, u)

Three possible inputs (r, d, n)

→ Nine possible transfer functions !

$$x = \frac{P}{1+P\cdot C}\cdot d + \frac{P\cdot C}{1+P\cdot C}\cdot n + \frac{P\cdot C\cdot F}{1+P\cdot C}\cdot r$$

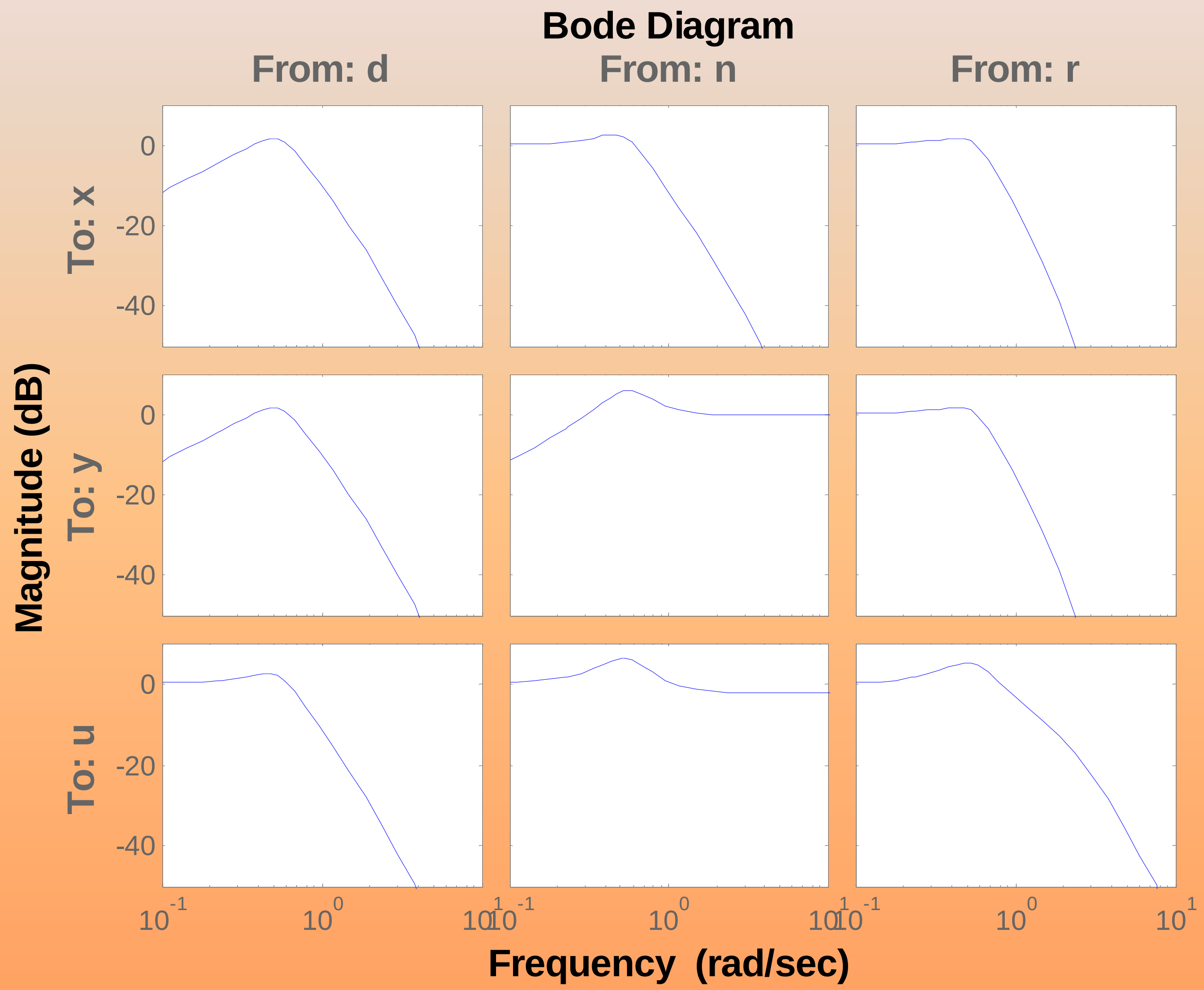
$$y = \frac{P}{1+P\cdot C}\cdot d + \frac{1}{1+P\cdot C}\cdot n + \frac{P\cdot C\cdot F}{1+P\cdot C}\cdot r$$

$$u = \frac{P\cdot C}{1+P\cdot C}\cdot d + \frac{C}{1+P\cdot C}\cdot n + \frac{C\cdot F}{1+P\cdot C}\cdot r$$

→ Six distinct transfer functions…

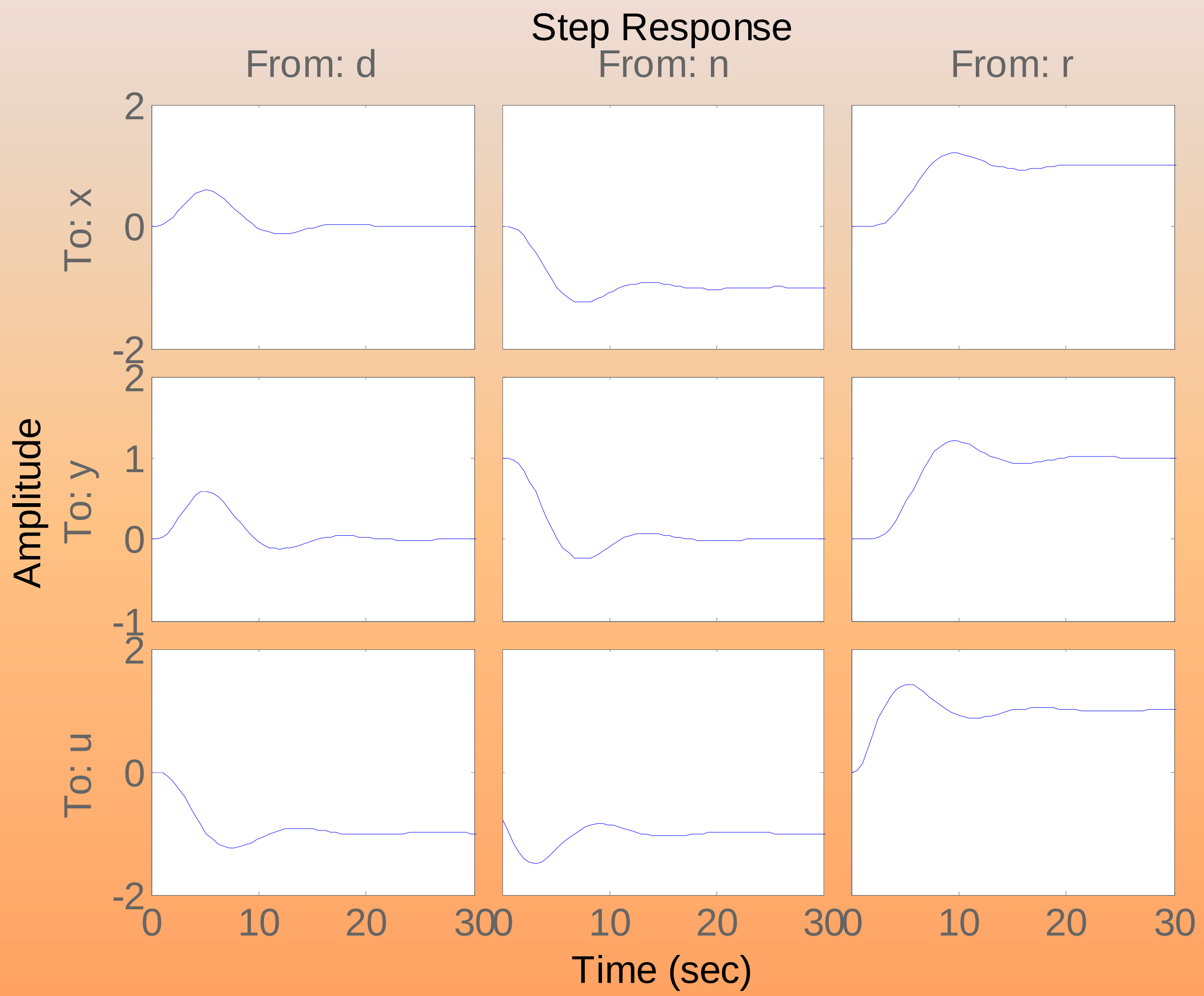
Relations between signals

→ Nine frequency responses…



Relations between signals

→ Nine step responses…



Relations between signals

A correct design means that each transfer has to be evaluated…

→ Need to be a little bit organized !

→ Need less criteria

  → Concept of sensibility functions

Sensibility functions

$$x = \frac{P}{1+P\cdot C}\cdot d + \frac{P\cdot C}{1+P\cdot C}\cdot n + \frac{P\cdot C\cdot F}{1+P\cdot C}\cdot r$$

$$y = \frac{P}{1+P\cdot C}\cdot d + \frac{1}{1+P\cdot C}\cdot n + \frac{P\cdot C\cdot F}{1+P\cdot C}\cdot r$$

$$u = \frac{P\cdot C}{1+P\cdot C}\cdot d + \frac{C}{1+P\cdot C}\cdot n + \frac{C\cdot F}{1+P\cdot C}\cdot r$$

$$L = P\cdot C$$

Loop sensitivity function

$$S = \frac{1}{1+L} = \frac{1}{1+P\cdot C}$$

Sensibility function

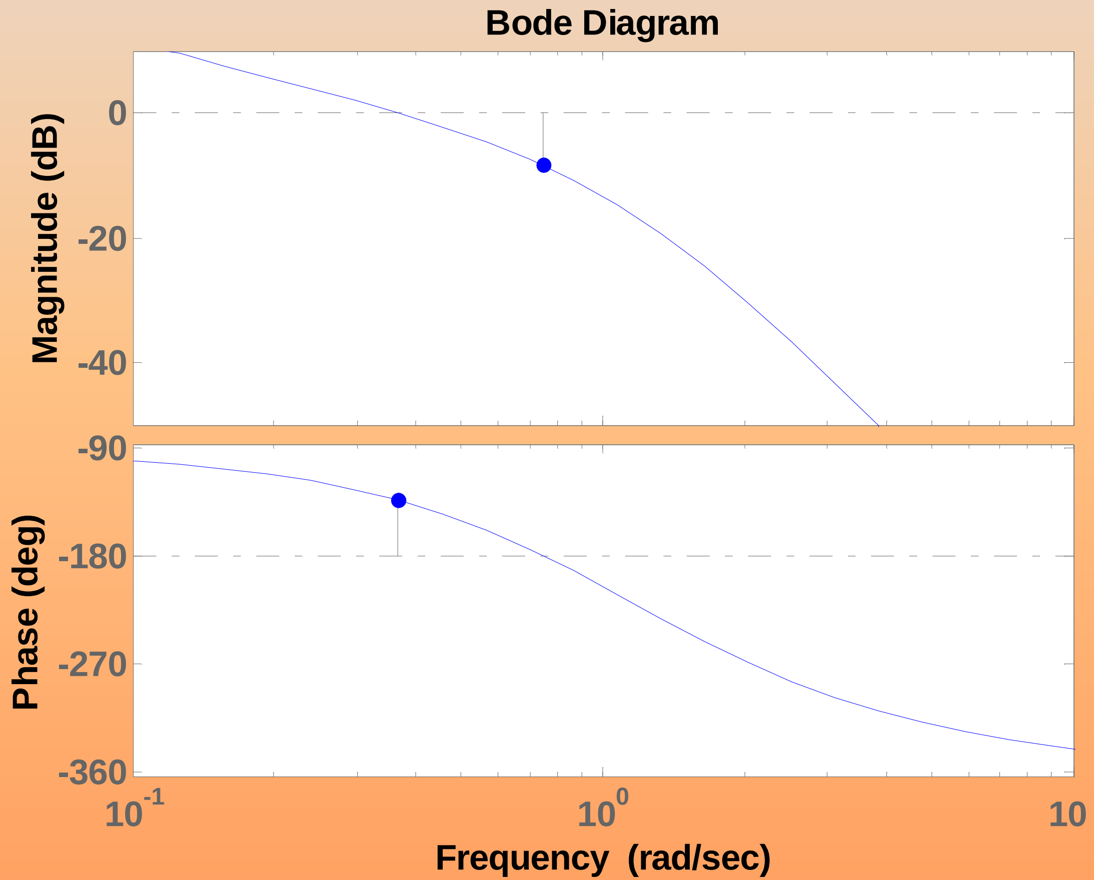
$$T = \frac{L}{1+L} = \frac{P\cdot C}{1+P\cdot C}$$

Complementary sensibility function

L tells everything about stability : common denominator of each transfer functions

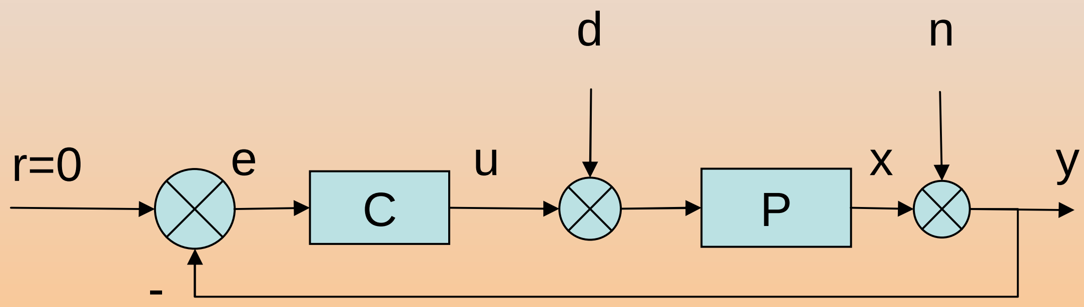
Sensibility functions

L=PC tells everything about stability : common denominator of each transfer functions



Sensibility functions

S=1/(1+L) tells about noise reduction



Without feedback :

$$y_{ol} = n + P \cdot d$$

With feedback control :

$$y_{cl} = \frac{1}{1+P \cdot C} n + \frac{P}{1+P \cdot C} \cdot d = S \cdot y_{ol}$$

→ Disturbances with $|S(i\omega)| < 1$ are reduced by feedback

→ Disturbances with $|S(i\omega)| > 1$ are amplified by feedback

Sensibility functions

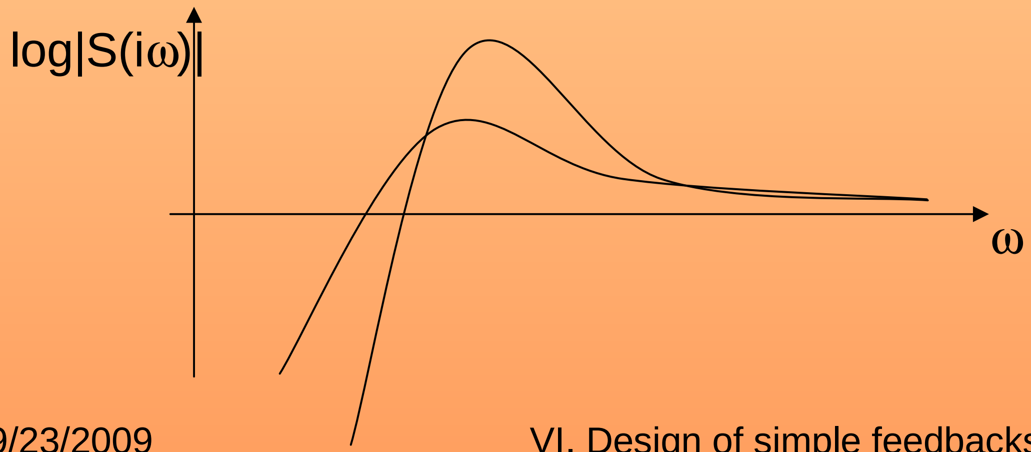
It would be nice to have $|S(i\omega)| < 1$ for all frequencies !

Cauchy Integral Theorem :

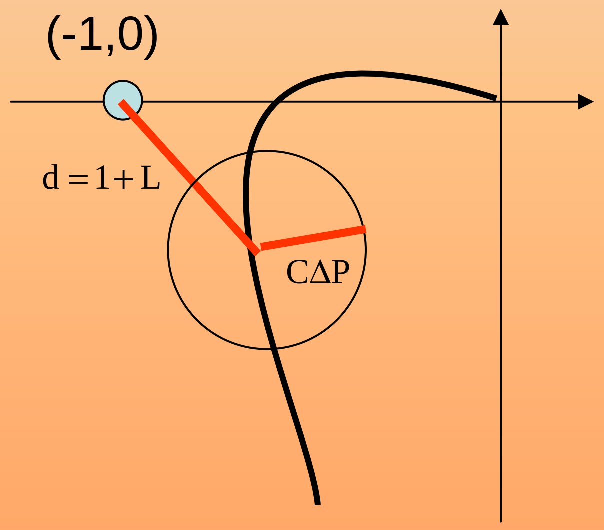
→ for stable open loop system : $\int_0^\infty \log|S(i\cdot\omega)| = 0$

→ For unstable or time delayed systems : $\int_0^\infty \log|S(i\cdot\omega)| > 0$

Conclusion : water bed effect…



Sensibility functions



Nyquist stability criteria :

$$\forall \omega, |C\Delta P| < |1+L|$$

$$\Leftrightarrow$$

$$\frac{|\Delta P|}{|P|} < \frac{1}{|T|}$$

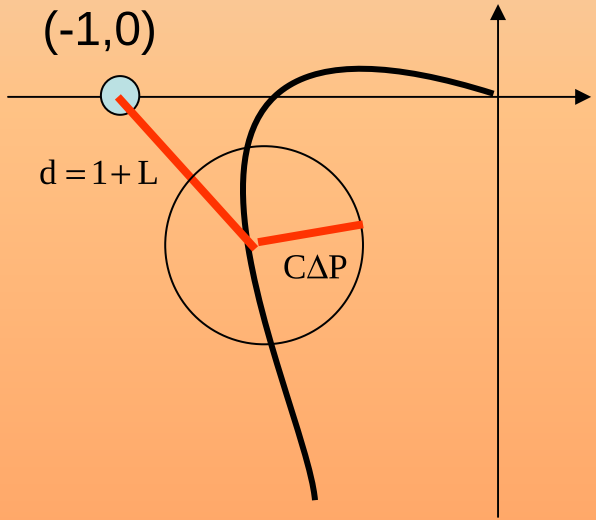
➔ 1/T tells how much P is allowed to vary until system becomes unstable

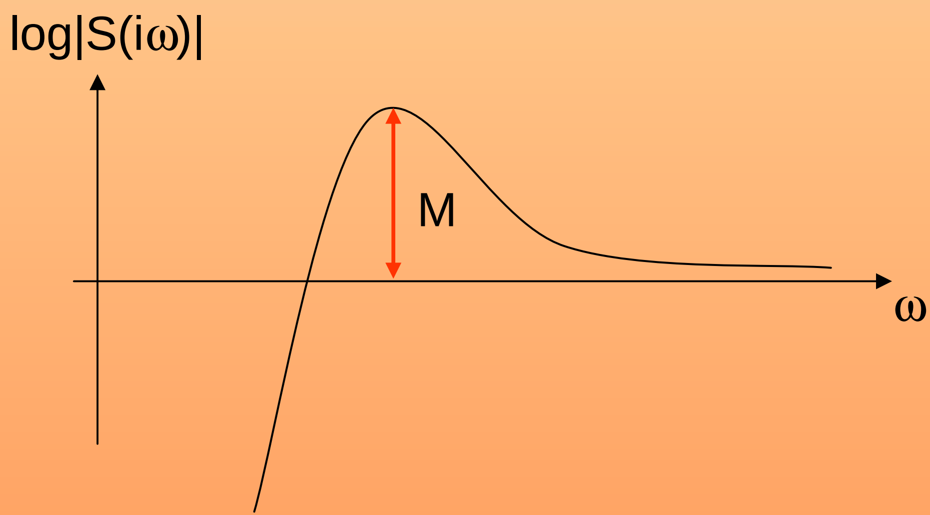
Sensibility functions

Nyquist stability criteria :

→ Minimum value of d tells how close of instability is the system

→ d_{min} is a measure of robustness : the bigger is M=1/d the more robust is the system





VII. Feedforward design

Introduction

Feedforward is a useful complement to feedback. Basic properties are:

+ Reduce effects of disturbance that can be measured

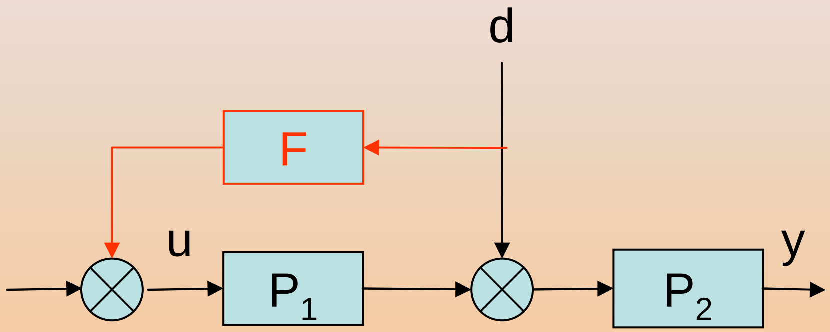
+ Improve response to reference signal

+ No risk for instability

- Design of feedforward is simple but requires good model and/or measurements

+ Beneficial when combined with feedback

Attenuation of measured disturbance



$$\frac{Y}{D} = P_2 \cdot (1 - P_1 \cdot F)$$

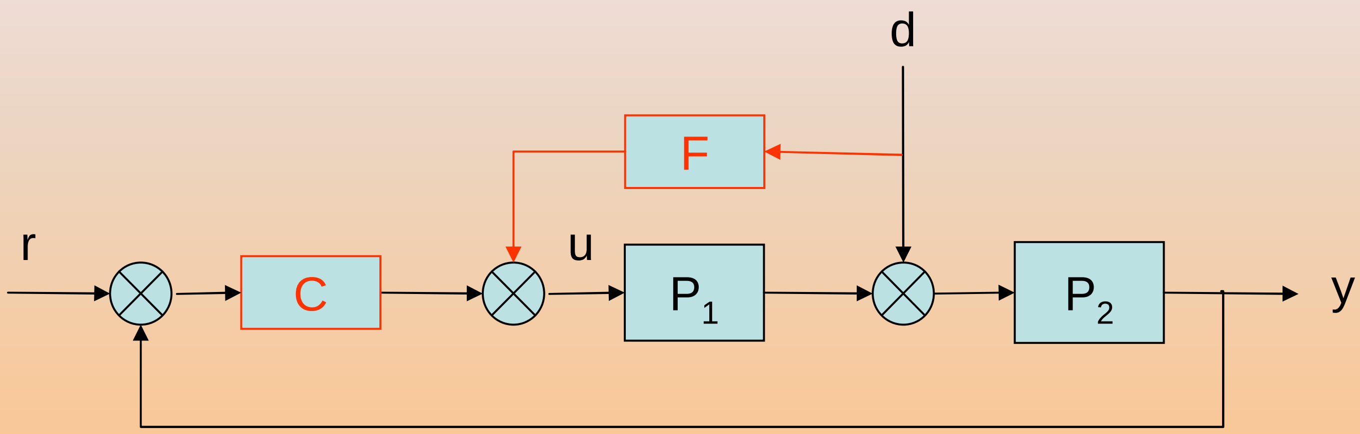
Disturbance is eliminated if F is chosen such as:

$$F = P_1^{-1}$$

→ Need to measure d

→ P1 needs to be inversible

Combined Feedback and Feedforward



Disturbance d is attenuated both by F and C :

$$\frac{Y}{D} = \frac{P_2 \cdot (1 - P_1 \cdot F)}{1 + P \cdot C}$$

System inverse

The ideal feedforward needs to compute the inverse of P_1. That's might be tricky… Examples:

$$P(s)=\frac{1}{1+s} \Rightarrow F(s)=P^{-1}(s)=1+s$$ Differentiation ☹

$$P(s)=\frac{e^{-s}}{1+s} \Rightarrow F(s)=P^{-1}(s)=(1+s)\cdot e^{s}$$ Prediction ☹

$$P(s)=\frac{1-s}{1+s} \Rightarrow F(s)=P^{-1}(s)=\frac{1+s}{1-s}$$ Unstable ☹

Approximate system inverse

The ideal feedforward needs to compute the inverse of P_1. That's might be tricky… Examples:

$$P(s) = \frac{1}{1+s} \quad \Rightarrow \quad F(s) = P^{-1}(s) = 1+s$$ Differentiation ☹

$$P(s) = \frac{e^{-s}}{1+s} \quad \Rightarrow \quad F(s) = P^{-1}(s) = (1+s) \cdot e^{s}$$ Prediction ☹

$$P(s) = \frac{1-s}{1+s} \quad \Rightarrow \quad F(s) = P^{-1}(s) = \frac{1+s}{1-s}$$ Unstable ☹

Approximate system inverse

Since it is difficult to obtain an exact inverse we have to approximate. One possibility is to find the transfer function which minimizes :

$$J = \int_0^{\infty} \left(u(t) - v(t)\right) \cdot dt$$

Where:

$$V = P \cdot X \cdot U$$

And where U is a particular input (ex: a step signal). This gives for instance:

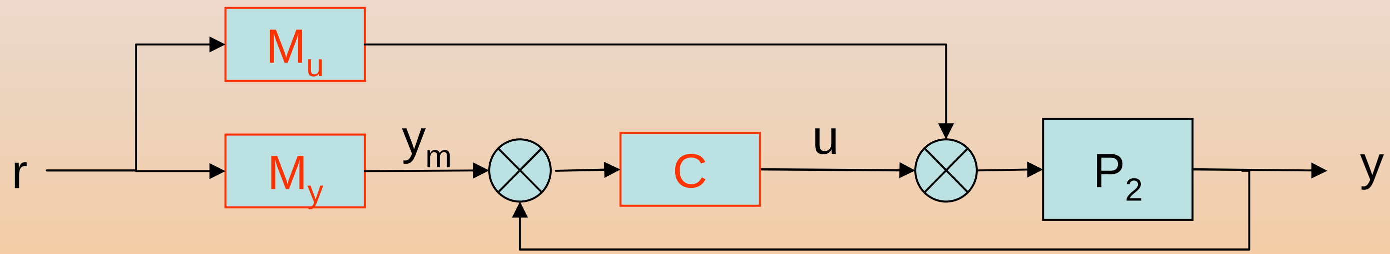
$$P(s) = \frac{1}{1+s} \quad \rightarrow \quad P^{-1}(s) \approx \frac{1+s}{1+T\cdot s}$$

$$P(s) = e^{-s} \quad \rightarrow \quad P^{-1}(s) \approx 1$$

$$P(s) = \frac{1-s}{1+s} \quad \rightarrow \quad F(s) = P^{-1}(s) = 1$$

Improved response to reference signal

The reference signal can be injected after the controller:



y_m is the desired trajectory.

Choose $M_u = M_y / P$

Design concerns:

→ M_u approximated

→ M_y adapted such that M_y/P feasible

Combining feedback and feedforward

Feedback
→ Closed loop
→ Acts only when there are deviations
→ Market driven
→ Robust to model errors
→ Risk for instability

Feedforward
→ Open loop
→ Acts before deviation shows up
→ Planning
→ Not robust to model errors
→ No risk for instability

➔ Feedforward must be used as a complement to feedback. Requires good modeling.

VIII. State feedback

Introduction

- Simple design becomes difficult for high order systems
- What is the *State* concept ?
  - State are the variables that fully summarize the actual state of the system
  - Future can be fully predicted from the current state
  - *State* is the ideal basis for control

State feedback

Let us suppose the system is described by the following equation (x is a vector, A, B and C are matrixes) :

$$\begin{cases} \frac{dx}{dt} = A \cdot x + B \cdot u \\ y = C \cdot x \end{cases}$$

The general linear controller is :

$$u = -K \cdot x + L \cdot u$$

The closed loop system then becomes :

$$\begin{cases} \frac{dx}{dt} = A \cdot x + B \cdot \left(-K \cdot x + L \cdot u\right) = \left(A - B \cdot K\right) \cdot x + B \cdot L \cdot u \\ y = C \cdot x \end{cases}$$

The closed loop system has the characteristic equation:

$$P\left(s\right) = \det\left(s \cdot I - \left(A - B \cdot K\right)\right)$$

State feedback

Let us suppose the system is described by the following equation (x is a vector, A, B and C are matrixes) :

$$\begin{cases} \frac{dx}{dt} = A \cdot x + B \cdot u \\ y = C \cdot x \end{cases}$$

Main mathematical tool is linear algebra and matrixes !

The general linear controller is :

$$u = -K \cdot x + L \cdot u$$

The closed loop system then becomes :

$$\begin{cases} \frac{dx}{dt} = A \cdot x + B \cdot \left(-K \cdot x + L \cdot u\right) = \left(A - B \cdot K\right) \cdot x + B \cdot L \cdot u \\ y = C \cdot x \end{cases}$$

The closed loop system has the characteristic equation:

$$P\left(s\right) = \det\left(s \cdot I - \left(A - B \cdot K\right)\right)$$

Pole placement

Original (open loop) system behavior depends on its poles, solution of the characteristic equation:

$$P_{OL}(s) = \det(s \cdot I - A)$$

Closed loop system behavior depends on its poles, solution of the characteristic equation:

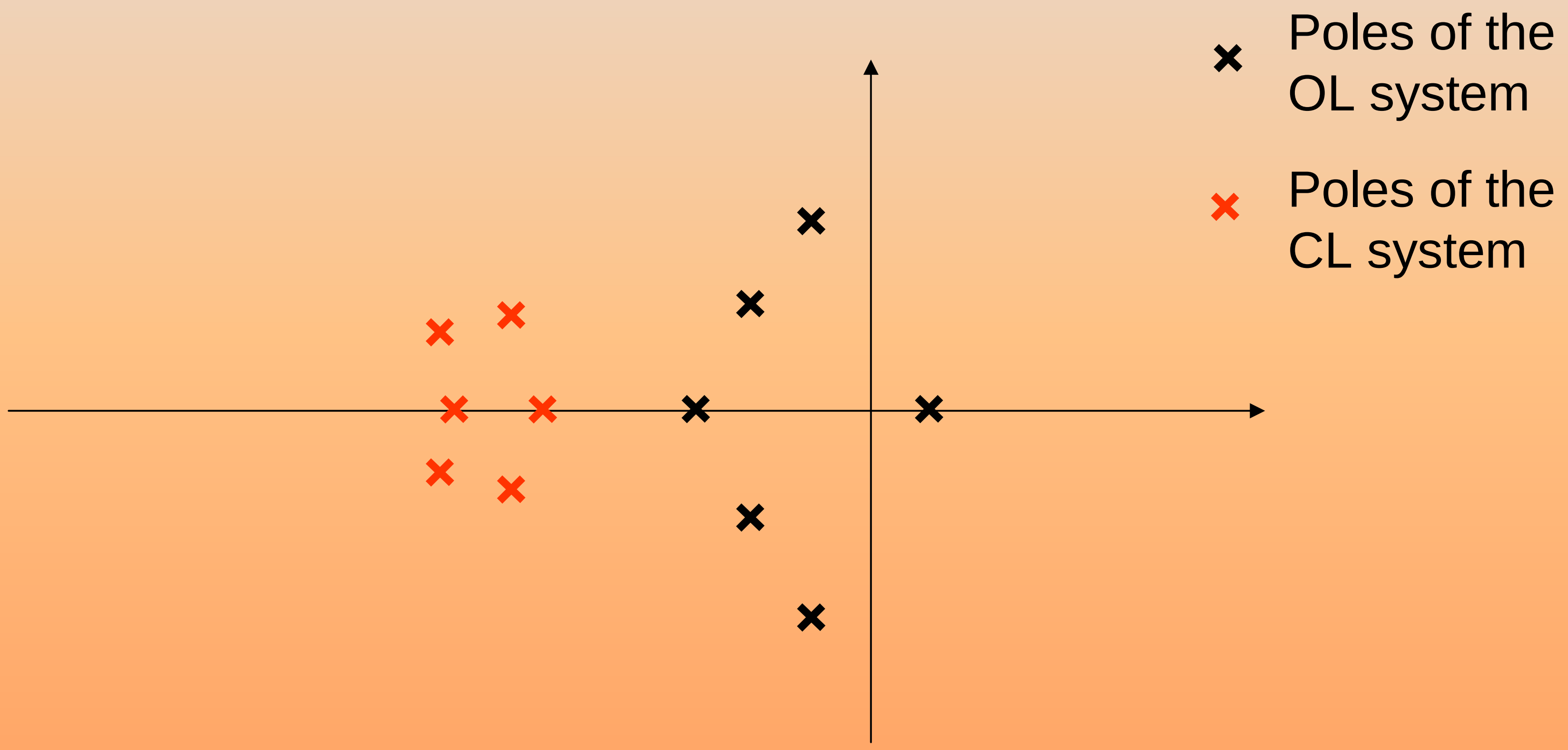
$$P_{CL}(s) = \det(s \cdot I - (A - B \cdot K))$$

Needs to tune N parameters (N : dimension of x and K)

Appropriate choice of K allow to place the poles anywhere ! (Needs simple mathematical skills (not detailed here ☺)

Two problems : observability, controllability

Pole placement



First problem : observability

In the control feedback equation x is supposed to be known. If one can access (measure) x, there is no problem. Sometimes, x cannot be measured but can be *observed*.

System described by:

$$\begin{cases} \dfrac{dx}{dt} = A \cdot x + B \cdot u \\ y = C \cdot x \end{cases}$$

Only u and y accessible, A and B known. Solution is to estimate internal state x with a "state observer" of gain K_o :

$$\begin{cases} \dfrac{dx_{obs}}{dt} = A \cdot x_{obs} + B \cdot u + K_{obs} \cdot \left(y - y_{obs} \right) \\ y_{obs} = C \cdot x_{obs} \end{cases}$$

Appropriate choice of K_{obs} minimizes $y_{obs} - y$: x_{obs} tends to x

Poles of the observer are the poles of: $P\left(s\right) = \det\left(s \cdot I - \left(A - K_{obs} \cdot C\right)\right)$

First problem : observability

In the control feedback equation x is supposed to be known. If one can access (measure) x, there is no problem. Sometimes, x cannot be measured but can be *observed*.

System described by:

$$\begin{cases} \dfrac{dx}{dt} = A \cdot x + B \cdot u \\ y = C \cdot x \end{cases}$$

Only u and y accessible, A and B known. Solution is to estimate internal state x with a "state observer" of gain K_o :

$$\begin{cases} \dfrac{dx_{obs}}{dt} = A \cdot x_{obs} + B \cdot u + K_{obs} \cdot (y - y_{obs}) \\ y_{obs} = C \cdot x_{obs} \end{cases}$$

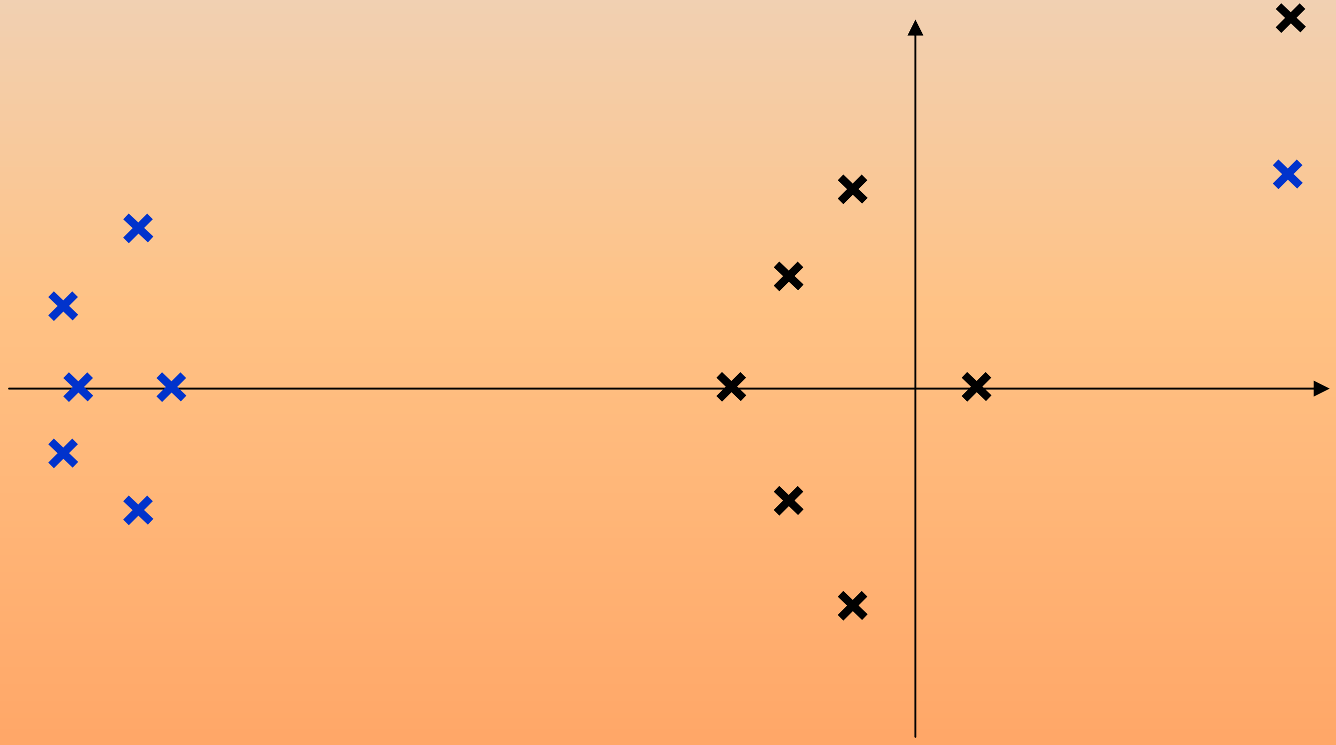
Appropriate choice of K_{obs} minimizes $y_{obs} – y$: x_{obs} tends to x

Poles of the observer are the poles of: $P(s) = \det\left(s \cdot I - \left(A - K_{obs} \cdot C\right)\right)$

First problem : observability

Poles of the observer are the poles those of:

$$P(s) = \det\left(s \cdot I - \left(A - K_{obs} \cdot C\right)\right)$$



Poles of the system

Poles of the observer

First problem : observability

Problem : is the system observable ?

In most cases : yes

Sometimes, the state is not observable :

→ The observer does not converge to the true state, whatever K_{obs} is.

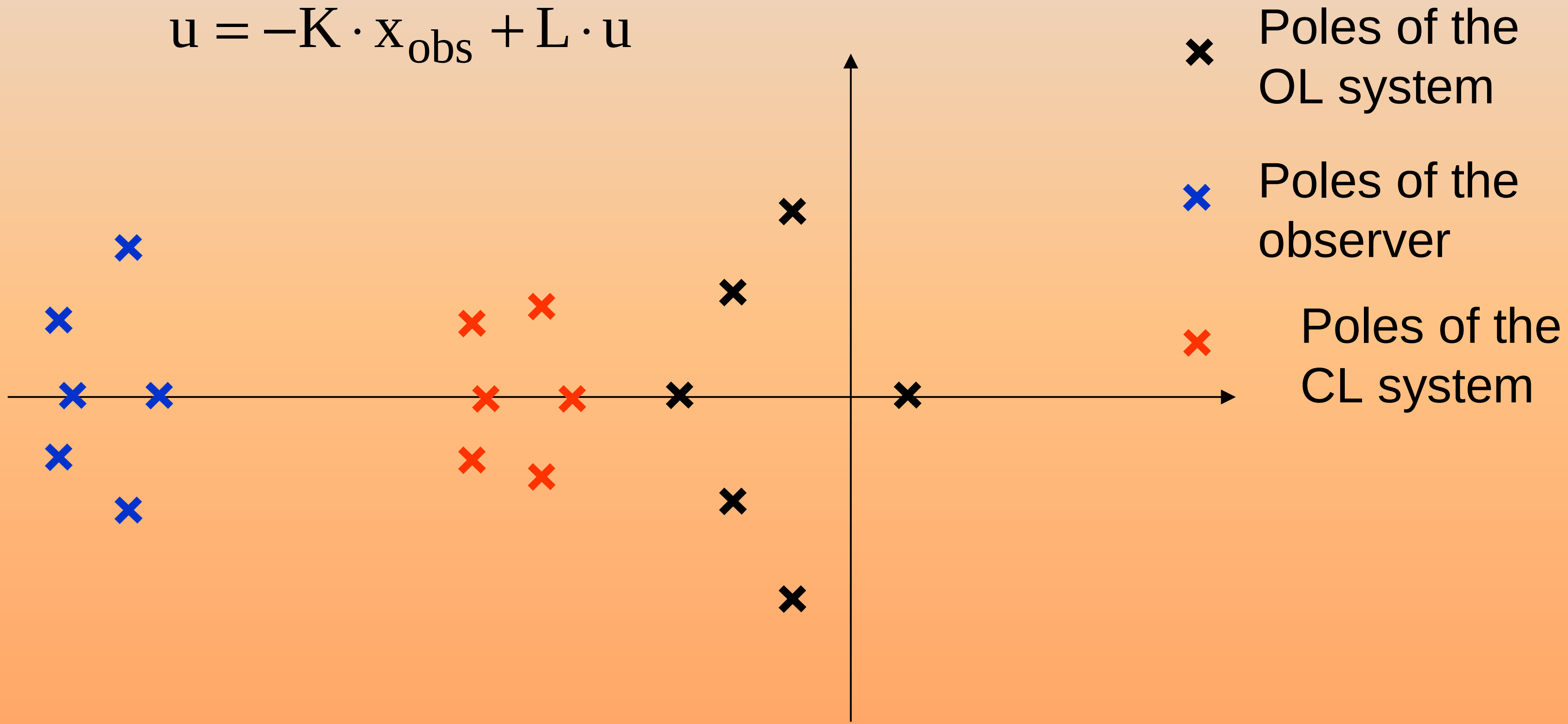
→ Can be derived from a mathematical analyses of (A,C):

$$rank(A, AC, AAC, AAAC\ldots) = N$$

Combining an observer and a state feedback

True (with x) state feedback can be replaced by an observed (x_{obs}) state feedback:

$$u = -K \cdot x_{obs} + L \cdot u$$



Second problem : controllability

Sometimes a state is not *controllable* : means that whatever the command u is, some parts of the state are not controllable

→ Can be derived from a mathematical analyses of (A,B):

$$rank(A,AB,AAB,AAAB\ldots) = N$$

Problem if :

- A state is not controllable and unstable
- A state is not controllable and slow

No problem if :

- A state is not controllable and fast (decays rapidly)