Controllability:

Contents

1. General result:

A linear continuous-time time-invariant system: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ is controllable iff: for any initial value $\mathbf{x}_0$ of the state at the initial time $t_0 = 0$ and for a given final time t_f, there exists a control signal trajectory $\mathbf{u}([0 \;\; t_f])$ such that $\mathbf{x}(t_f) = \mathbf{0}$.

Let n be the system order ($\mathbf{x} \in \mathbb{R}^n$). A necessary and sufficient condition for the system to be **controllable** (condition **C1**) is:

$$\mathrm{rank}(\mathcal{C}) = n \;\text{ with: } \mathcal{C} = \begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix} \;\; (\text{condition } \mathbf{C2})\,.$$

$\mathcal{C}$ is called the **controllability matrix** .

2. Proof ($\mathbf{C1} \Leftrightarrow \mathbf{C2}$):

The general solution of the state equation is:

$$\mathbf{x}(t_f) = \mathrm{e}^{\mathbf{A}(t_f - t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t_f} \mathrm{e}^{\mathbf{A}(t_f - \tau)}\mathbf{B}\mathbf{u}(\tau)d\tau.$$

Applied to our problem ($t_0 = 0$, $\mathbf{x}(t_0) = \mathbf{x}_0$, $\mathbf{x}(t_f) = \mathbf{0}$), it comes:

$$\mathbf{0} = \mathrm{e}^{\mathbf{A}t_f}\mathbf{x}_0 + \int_0^{t_f} \mathrm{e}^{\mathbf{A}(t_f - \tau)}\mathbf{B}\mathbf{u}(\tau)d\tau.$$

or:

$$\int_0^{t_f} \mathrm{e}^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau = -\mathbf{x}_0 \;\; (1)\,.$$

3. Proof of the necessary condition ($\overline{\mathbf{C2}} \Rightarrow \overline{\mathbf{C1}}$):

Cayley-Hamilton theorem (see: `Cayley Hamilton`) allows to express:

$$\mathrm{e}^{-\mathbf{A}\tau} = \sum_{i=0}^{n-1} g_i(-\tau)(-1)^i \tau^i \mathbf{A}^i = \sum_{i=0}^{n-1} \widetilde{g}_i(\tau)\mathbf{A}^i \quad (2)\ .$$

Considering, without loss of generality a single input system ($\mathbf{u}(t) \rightarrow u(t)$), equation (1) can be rewritten as :

$$\sum_{i=0}^{n-1} \mathbf{A}^i\mathbf{B} \underbrace{\int_0^{t_f} \widetilde{g}_i(\tau)u(\tau)d\tau}_{h_i(u([0\ \ t_f]))} = -\mathbf{x}_0\ .$$

or:

$$\underbrace{\left[\mathbf{B}\ \ \mathbf{AB}\ \ \mathbf{A}^2\mathbf{B}\ \cdots\ \mathbf{A}^{n-1}\mathbf{B}\right]}_{\mathcal{C}\ \ (n\times n)} \underbrace{\begin{bmatrix} h_0(u([0\ \ t_f])) \\ h_1(u([0\ \ t_f])) \\ \vdots \\ h_{n-1}(u([0\ \ t_f])) \end{bmatrix}}_{\mathbf{h}(u([0\ \ t_f]))\ \ (n\times 1)} = -\underbrace{\mathbf{x}_0}_{(n\times 1)} \quad (3)\ .$$

If rank$(\mathcal{C}) < n$ ($\overline{\mathbf{C2}}$) then there exists a vector $\mathbf{v} \neq \mathbf{0}_{n\times 1}$ such that $\mathbf{v}^T\mathcal{C} = \mathbf{0}$. Then pre-multiplying (3) by $\mathbf{v}^T$, it comes:

$$\mathbf{v}^T\ \mathcal{C}\ \mathbf{h}(u([0\ \ t_f])) = 0 = -\mathbf{v}^T\mathbf{x}_0\ .$$

Thus the system (3) is undetermined and it is not possible de compute $\mathbf{h}(u([0\ \ t_f]))$ and thus it is not possible to compute $u([0\ \ t_f])$ to bring back the state to $\mathbf{0}$ at t_f (**then** the system is **uncontrollable** ($\overline{\mathbf{C1}}$)) except if $\mathbf{x}_0$ is in the controllolability sub-space $\mathcal{S}_c$ characterized by:

$$\mathcal{S}_c = \{\mathbf{x} \in \mathbb{R}^n\ /\ \mathbf{v}^T\mathbf{x} = 0,\ \forall \mathbf{v} \in \mathrm{null}(\mathcal{C})\}\ .$$

where null$(\mathcal{C})$ is the null-space of the matrix $\mathcal{C}$.

4. Proof of the sufficient condition ($\mathbf{C2} \Rightarrow \mathbf{C1}$):

For the sufficient condition, a new condition **C3** is introduced:

$$\mathbf{Q}_c(t_f) = \int_0^{t_f} \mathrm{e}^{-\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T\mathrm{e}^{-\mathbf{A}^T\tau}d\tau \ \text{ is an invertible } n\times n \text{ matrix (condition } \mathbf{C3})\ .$$

Then **C3** $\Rightarrow$ **C1**. Indeed, the particular control signal:

$$\mathbf{u}(t) = -\mathbf{B}^T\mathrm{e}^{-\mathbf{A}^Tt}\mathbf{Q}_c^{-1}(t_f)\mathbf{x}_0 \quad (4)$$

satisfies the problem (equation (1)). And thus the system is **controllable** (**C1**).

(Indeed: (3) in (1) gives: $-\int_0^{t_f} \mathrm{e}^{-\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T\mathrm{e}^{-\mathbf{A}^T\tau}\mathbf{Q}_c^{-1}(t_f)\mathbf{x}_0 d\tau = -\mathbf{Q}_c(t_f)\mathbf{Q}_c^{-1}(t_f)\mathbf{x}_0 = -\mathbf{x}_0$.)

To prove **C2** $\Rightarrow$ **C3**, it can be proven that $\overline{\mathbf{C3}} \Rightarrow \overline{\mathbf{C2}}$:
if $\mathbf{Q}_c(t_f)$ is not invertible ($\overline{\mathbf{C3}}$), then considering a vector $\mathbf{v} \neq \mathbf{0}_{n\times 1} \in$ null($\mathbf{Q}_c(t_f)$), it comes:

$$\mathbf{v}^T\mathbf{Q}_c(t_f)\mathbf{v} = 0 = \int_0^{t_f} \underbrace{\mathbf{v}^T\mathrm{e}^{-\mathbf{A}\tau}\mathbf{B}\,\underbrace{\mathbf{B}^T\mathrm{e}^{-\mathbf{A}^T\tau}\mathbf{v}}_{\mathbf{p}_v(\tau)}}_{q_v(\tau)=\mathbf{p}_v^T(\tau)\mathbf{p}_v(\tau)} d\tau \ .$$

Remark : in the single input case: $\mathbf{p}_v^T(\tau) = \mathbf{p}_v(\tau) = p_v(\tau)$ and $q_v(\tau) = p_v^2(\tau)$.

$q_v(\tau) = \mathbf{p}_v^T(\tau)\mathbf{p}_v(\tau)$ is a scalar quadratic function of τ and thus is always positive or null ($\forall\ \tau$). For its intregral to be null, it requires that: $\mathbf{p}_v^T(\tau) = 0,\ \forall\ \tau \in [0\ \ t_f]$. Thus:

$$\overline{\mathbf{C3}} \Rightarrow \mathbf{v}^T\mathrm{e}^{-\mathbf{A}\tau}\mathbf{B} = 0,\ \forall\ \tau \in [0\ \ t_f]$$

or using the **Cayley-Hamilton** expresion of $\mathrm{e}^{-\mathbf{A}\tau}$ (equation (2)):

$$\overline{\mathbf{C3}} \Rightarrow \mathbf{v}^T \sum_{i=0}^{n-1} \widetilde{g}_i(\tau)\mathbf{A}^i\mathbf{B} = 0,\ \ \forall\ \tau \in [0\ \ t_f]$$

$$\Leftrightarrow \mathbf{v}^T \underbrace{\left[\mathbf{B}\ \ \mathbf{AB}\ \ \mathbf{A}^2\mathbf{B}\ \cdots\ \mathbf{A}^{n-1}\mathbf{B}\right]}_{\mathcal{C}\ \ (n\times n)} \underbrace{\begin{bmatrix} \widetilde{g}_0(\tau) \\ \widetilde{g}_1(\tau) \\ \vdots \\ \widetilde{g}_{n-1}(\tau) \end{bmatrix}}_{\widetilde{\mathbf{g}}(\tau)} = 0,\ \ \forall\ \tau \in [0\ \ t_f]$$

$$\Leftrightarrow \mathbf{v}^T\,\mathcal{C} = \mathbf{0}_{1\times n}$$

$$\Leftrightarrow \mathrm{rank}(\mathcal{C}) < n\ \ (\overline{\mathbf{C2}})$$

Thus: **C2** $\Rightarrow$ **C3** $\Rightarrow$ **C1** $\bullet$

5. Discrete-time case:

A linear discrete-time time-invariant system: $\mathbf{x}_{k+1} = \mathbf{A}_d\mathbf{x}_k + \mathbf{B}_d\mathbf{u}_k$ is controllable iff: for any initial value $\mathbf{x}_0$ of the state at the initial time $k = 0$ and for a given final time N, there exists a control signal sequence $[\mathbf{u}_0,\ \mathbf{u}_1, \cdots, \mathbf{u}_{N-1}]^T$ such that $\mathbf{x}_N = \mathbf{0}$.

Let n be the system order ($\mathbf{x} \in \mathbb{R}^n$). A necessary and sufficient condition for the system to be **controllable** (condition **C1d**) is:

$$\mathrm{rank}(\mathcal{C}_d) = n\ \ \text{with: } \mathcal{C}_d = \left[\mathbf{B}_d\ \ \mathbf{A}_d\mathbf{B}_d\ \ \mathbf{A}_d^2\mathbf{B}_d\ \cdots\ \mathbf{A}_d^{n-1}\mathbf{B}_d\right]\ \ (\text{condition } \mathbf{C2d})\ .$$

$\mathcal{C}_d$ is called the **controllability matrix** . **Proof:** the proof in the discrete-time case is simpler than in the continuous-time case. Note that in the discrete-time case, there is a constraint on the final time: $N \geq n$. Indeed, the direct integration of the state equation with $N = n$ reads:

$$\mathbf{x}_1 = \mathbf{A}_d\mathbf{x}_0 + \mathbf{B}_d\mathbf{u}_0$$

$$\mathbf{x}_2 = \mathbf{A}_d\mathbf{x}_1 + \mathbf{B}_d\mathbf{u}_1 = \mathbf{A}_d^2\mathbf{x}_0 + [\mathbf{B}_d \;\; \mathbf{A}_d\mathbf{B}_d]\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_0\end{bmatrix}$$

$$\vdots$$

$$\mathbf{x}_n = \mathbf{A}_d^n\mathbf{x}_0 + \underbrace{[\mathbf{B}_d \;\; \mathbf{A}_d\mathbf{B}_d \;\; \cdots \mathbf{A}_d^{n-1}\mathbf{B}_d]}_{\mathcal{C}_d}\begin{bmatrix}\mathbf{u}_{n-1}\\ \vdots \\ \mathbf{u}_1\\ \mathbf{u}_0\end{bmatrix} = \mathbf{0}$$

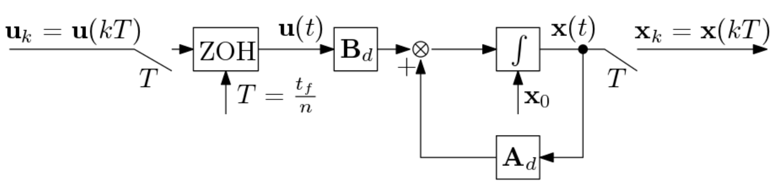
Thus, one can solve this last equation in $[\mathbf{u}_{n-1} \cdots \mathbf{u}_1 \quad \mathbf{u}_0]^T$ if and only if $\text{rank}(\mathcal{C}_d) = n$. Then the solution reads:

$$\begin{bmatrix}\mathbf{u}_{n-1}\\ \vdots \\ \mathbf{u}_1\\ \mathbf{u}_0\end{bmatrix} = -\mathcal{C}_d^{-1}\,\mathbf{A}_d^n\,\mathbf{x}_0 \quad (5) \qquad \bullet$$

6. A particular solution in the continuous-time case:

Considering the continuous-time system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ with an initial state $\mathbf{x}_0$, the objective is to find a control signal trajectory $\mathbf{u}([0 \;\; t_f])$ such that $\mathbf{x}(t_f) = \mathbf{0}$ for a given final time t_f.

The discrete-time case can provide a solution to this problem considering that $\mathbf{u}(t)$ is the output of a zero order hold (ZOH) sampled at $T = t_f/n$ (see the following figure):



The general solution of the continuous-time state equation:

$$\mathbf{x}(t_f) = \mathrm{e}^{\mathbf{A}(t_f - t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t_f} \mathrm{e}^{\mathbf{A}(t_f-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau\,,$$

with: $t_0 = kT$, $\mathbf{x}(t_0) = \mathbf{x}_k$, $t_f = (k+1)T$, $\mathbf{x}(t_f) = \mathbf{x}_{k+1}$, $\mathbf{u}(\tau) = \mathbf{u}_k$, $\forall\ \tau \in [kT,\ \ (k+1)T[$ and the change of variable $v = (k+1)T - \tau$ gives the discrete-time state-space representation of the system between $\mathbf{u}_k$ and $\mathbf{x}_k$:

$$\mathbf{x}_{k+1} = \underbrace{\mathrm{e}^{\mathbf{A}T}}_{\mathbf{A}_d}\ \mathbf{x}_k + \underbrace{\int_0^T \mathrm{e}^{\mathbf{A}v}\mathbf{B}dv}_{\mathbf{B}_d}\ \mathbf{u}_k$$

Then the solution (5) can be computed and applied to the input of the zero order hold such that $\mathbf{x}_n = \mathbf{x}(nT) = \mathbf{x}(t_f) = \mathbf{0}$. Solution (5) provides $\mathbf{u}_0,\ \mathbf{u}_1,\ \cdots,\ \mathbf{u}_{n-1}$, of course: $\mathbf{u}_k = \mathbf{0}$, $\forall\ k \geq n$.

That is illustrated in the following MATLAB session for a 4-th order continuous-time system with an arbitrary initial condition and a given final time t_f.

7. Illustration:

Given:

- a continous-time system: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$,
- an arbitrary initial condition $\mathbf{x}_0$,
- a given final time t_f,

an open-loop control profile $u([0\ \ t_f])$ is computed such that $\mathbf{x}(t_f) = \mathbf{0}$:

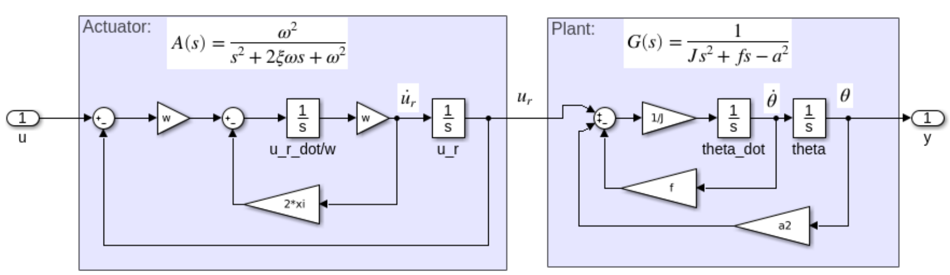
The proposed example is the serie-connection of:

- the plant model which corresponds to an unstable aerospace vehicule: $G(s) = \frac{1}{Js^2+fs-a^2}$,
- the actuator model: $A(s) = \frac{\omega^2}{s^2+2\xi\omega s+\omega^2}$.

and is described by a SIMILINK file:

Numerical application: $J = 1\,Kgm^2$, $f = 0.01\,Nms/rd$, $a^2 = 1\,Nm/rd$, $\omega = 10\,rd/s$, $\xi = 0.7$.

```
J=1;a2=1;f=0.01;xi=0.7;w=10;        % Numerical application.
open_system('model_4th_order');
```



```
sys=linmod('model_4th_order');
sys.StateName


ans =

  41 cell array

    {'model_4th_order/theta'     }
    {'model_4th_order/u_r_dot//w'}
    {'model_4th_order/u_r'       }
    {'model_4th_order/theta_dot' }
```

From now, we know that the state vector provided by `linmod` is: $\mathbf{x} = [\theta,\ \dot{u}_r/\omega,\ u_r,\ \dot{\theta}]^T$.

```
A=sys.a;B=sys.b;
n=size(A,1);
Gc=ss(A,B,eye(n),0);        % continuous-time system
tf=3;                       % final time
Gd=c2d(Gc,tf/n);            % discrete-time system
Ad=Gd.a;Bd=Gd.b;
Cd=Bd;                      % controllability matrix
for ii=1:n-1;
    Cd=[Ad*Cd(:,1) Cd];     % here : Cd=[Ad^(n-1)Bd,..., AdBd, Bd].
end
rank(Cd)                    % check the system is controllable


ans =

     4
```

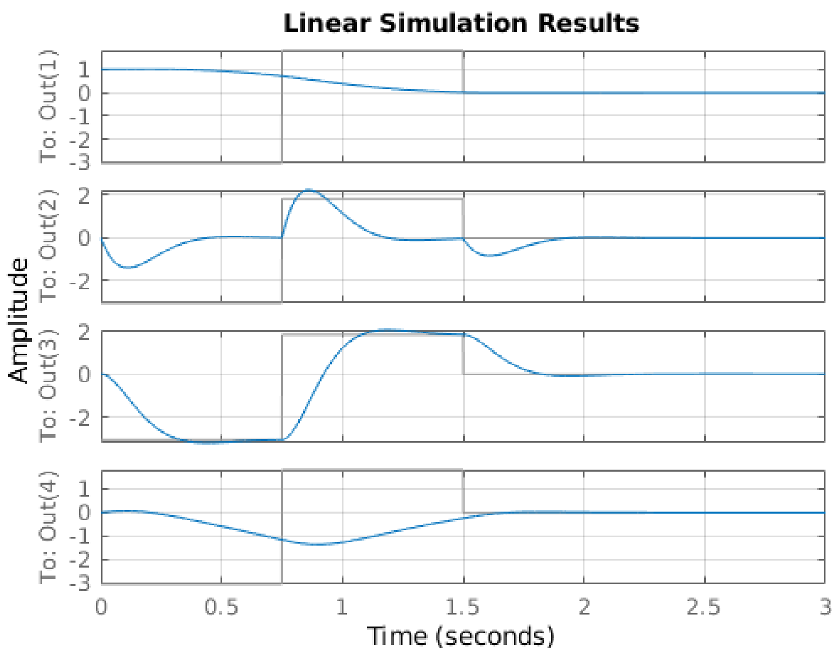
We consider an initial condition on the attitude θ:

```
x0=[1;0;0;0];               % initial condition
u_n=-inv(Cd)*Ad^n*x0        % [u_0, u_1, u_2, u_3]^T
%
% Simulation:
N=1000;
t=[0:tf/n/N:tf]';
U1=[];
for ii=1:n,
    U1=[U1;u_n(ii)*ones(N,1)];
```

```
end
U1=[U1;0];
figure
lsim(Gc,U1,t,x0,'zoh');
grid
```

```
u_n =

   -3.0464
    1.8214
   -0.0114
    0.0000
```



the stair-shape signal $u(t)$ due to the ZOH sampled at $t_f/4$ can be seen on the various plot (grey lines).

8. Another solution in continuous-time:

This solution is directly computed in continuous-time and is given by equation (4). Indeed $\mathbf{Q}_c(t_f) = \int_0^{t_f} \mathrm{e}^{-\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T\mathrm{e}^{-\mathbf{A}^T\tau}d\tau$ is the solution of the Lyapunov equation:

$$\mathbf{A}\mathbf{Q}_c(t_f) + \mathbf{Q}_c(t_f)\mathbf{A}^T - \mathbf{B}\mathbf{B}^T + \mathrm{e}^{-\mathbf{A}t_f}\mathbf{B}\mathbf{B}^T\mathrm{e}^{-\mathbf{A}^T t_f} = \mathbf{0}$$

Proof:

$$\mathbf{A}\mathbf{Q}_c(t_f) + \mathbf{Q}_c(t_f)\mathbf{A}^T = \int_0^{t_f} \underbrace{\mathbf{A}\mathrm{e}^{-\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T\mathrm{e}^{-\mathbf{A}^T\tau} + \mathrm{e}^{-\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T\mathrm{e}^{-\mathbf{A}^T\tau}\mathbf{A}^T}_{\frac{-d\left(\mathrm{e}^{-\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T\mathrm{e}^{-\mathbf{A}^T\tau}\right)}{d\tau}} d\tau$$

$$\mathbf{A}\mathbf{Q}_c(t_f)+\mathbf{Q}_c(t_f)\mathbf{A}^T = -\left[\mathrm{e}^{-\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T\mathrm{e}^{-\mathbf{A}^T\tau}\right]_0^{t_f} = \mathbf{B}\mathbf{B}^T - \mathrm{e}^{-\mathbf{A}t_f}\mathbf{B}\mathbf{B}^T\mathrm{e}^{-\mathbf{A}^T t_f} \quad \bullet$$

Thus $\mathbf{Q}_c(t_f)$ can be easily computed using the MATLAB function `lyap`.

Then the control signal bringing the state to $\mathbf{0}$ at t_f ($\mathbf{x}(t_f) = \mathbf{0}$) is:

$$\mathbf{u}(t) = -\mathbf{B}^T\mathrm{e}^{-\mathbf{A}^T t}\mathbf{Q}_c^{-1}(t_f)\mathbf{x}_0\,, \;\; \forall\, t \in [0,\;\; t_f]$$

$$\mathbf{u}(t) = \mathbf{0}\,, \;\; \forall\, t > t_f\;.$$

That is illustrated in the following MATLAB sequence:

```
Qc_tf=lyap(A,-B*B'+expm(-A*tf)*B*B'*expm(-A'*tf))


Qc_tf =

   1.0e+19 *

    0.0001    0.0011    0.0075   -0.0012
    0.0011    0.4137   -0.4917    0.0270
    0.0075   -0.4917    1.5170   -0.1627
   -0.0012    0.0270   -0.1627    0.0201
```

Clearly $\mathbf{Q}_c(t_f)$ is ill-conditionned !! Such a solution is thus very sensitive to the condition number of $\mathbf{Q}_c(t_f)$ and works ony for very small values of t_f:

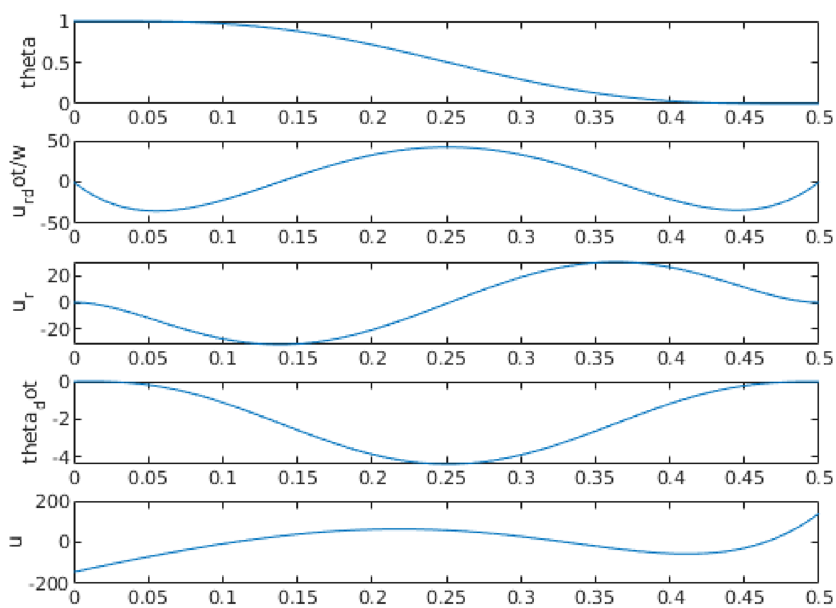
```
tf=0.5;
Qc_tf=lyap(A,-B*B'+expm(-A*tf)*B*B'*expm(-A'*tf))
Qc_tf_m1=inv(Qc_tf);
t=[0:tf/1000:tf];          % requires a very small step!!
U2=[];
for ii=1:length(t),
    U2=[U2;-B'*expm(-A'*t(ii))*Qc_tf_m1*x0];
end
Y=lsim(Gc,U2,t,x0);
figure
subplot(5,1,1)
plot(t,Y(:,1));ylabel('theta');
```

```
subplot(5,1,2);
plot(t,Y(:,2));ylabel('u_r_dot/w');
subplot(5,1,3)
plot(t,Y(:,3));ylabel('u_r');
subplot(5,1,4);
plot(t,Y(:,4));ylabel('theta_dot');
subplot(5,1,5);
plot(t,U2);ylabel('u');
```

```
Qc_tf =

   1.0e+03 *

    0.0019    0.1112    0.0024   -0.0123
    0.1112    8.1208   -1.8612   -0.5845
    0.0024   -1.8612    2.3109   -0.1542
   -0.0123   -0.5845   -0.1542    0.0863
```



One can see that the magnitude of the control signal (last plot) is very big ($\approx$ 200).

9. A closed-loop solution:

In the two previous illustrations, the control signal $u([0 \;\; t_f])$ is applied in open-loop. Although it meets the constraint $\mathbf{x}(t_f) = \mathbf{0}$, the system is still unstable for $t > t_f$. From a practical point of view a closed-loop solution is preferred.

Then, one can use a variation of the controllability definition (for linear system): a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ is controllable iff one can compute a state feedback assigning the n closed-loop eigenvalues to any prescribed values $\lambda_i,\ i = 1, \dots, n$. Of course one will choose these n eigenvalues stable and fast enough to bring back the state to $\mathbf{0}$ efficiently. But this is an exponential converge to $\mathbf{0}$, contrary to the initial definition. This new definition allows to introduce the

Controllability Canonical Form (CCF) $(\widetilde{\mathbf{A}}_c, \widetilde{\mathbf{B}}_c)$ of the linear system:

$$\text{CCF: } \dot{\widetilde{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & 0 & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}}_{\widetilde{\mathbf{A}}_c} \widetilde{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\widetilde{\mathbf{B}}_c} u$$

One can recognize:

- the coefficient a_i of the characteristic polynomial of the open-loop system ($\det(s\mathbf{I}_n - \mathbf{A})$) on the last row of the matrix $\widetilde{\mathbf{A}}_c$ (with a minus sign and by increasing order of power of s),
- the matrix $\widetilde{\mathbf{B}}_c$ with 0 every where except 1 in the last row.

The objective is thus to find a state feedback control law:

$$u = -\underbrace{[\widetilde{k}_0 \ \ \widetilde{k}_1 \ \ \cdots \ \ \widetilde{k}_{n-1}]}_{\widetilde{\mathbf{K}}} \widetilde{\mathbf{x}} = -\widetilde{\mathbf{K}}\widetilde{\mathbf{x}}$$

such that the closed-loop dynamics $(\widetilde{\mathbf{A}}_c - \widetilde{\mathbf{B}}_c\widetilde{\mathbf{K}})$ fits the prescribed dynamics:

$$\det(s\mathbf{I}_n - \widetilde{\mathbf{A}}_c + \widetilde{\mathbf{B}}_c\widetilde{\mathbf{K}}) = \Pi_{i=0}^{n}(s - \lambda_i) = \alpha_0 + \alpha_1 \mathrm{s} + \alpha_2 \mathrm{s}^2 + \cdots + \alpha_{n-1}\mathrm{s}^{n-1} + \mathrm{s}^n \ .$$

Then the solution is **obvious** :

$$\widetilde{k}_i = a_i - \alpha_i, \ \ \forall\, i = 0, 1, \cdots, n-1 \ .$$

$$\text{Indeed: } \widetilde{\mathbf{A}}_c - \widetilde{\mathbf{B}}_c\widetilde{\mathbf{K}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & 0 & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & \cdots & \cdots & 0 & 1 \\ \underbrace{-a_0 - \widetilde{k}_0}_{-\alpha_0} & \underbrace{-a_1 - \widetilde{k}_1}_{-\alpha_1} & \cdots & \underbrace{-a_{n-2} - \widetilde{k}_{n-2}}_{-\alpha_{n-2}} & \underbrace{-a_{n-1} - \widetilde{k}_{n-1}}_{-\alpha_{n-1}} \end{bmatrix} .$$

It is shown that an arbitrary linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ can be tansformed into the CCN by a regular change of state vector $\mathbf{x} = \mathbf{M}\widetilde{\mathbf{x}}$ iff:

$$\text{rank}(\mathcal{C}) = n \;\text{ with: } \mathcal{C} = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix} \quad (\text{condition } \mathbf{C2}) \; .$$

Proof: It is well known that: $\widetilde{\mathbf{A}}_c = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$ and $\widetilde{\mathbf{B}}_c = \mathbf{M}^{-1}\mathbf{B}$. We are going to identify $\mathbf{M}$ (column per column, starting from the last column) form the relations $\mathbf{M}\widetilde{\mathbf{A}}_c = \mathbf{A}\mathbf{M}$ and $\mathbf{M}\widetilde{\mathbf{B}}_c = \mathbf{B}$ and then check that $\mathbf{M}$ is invertible:

$$\underbrace{[\mathbf{m}_1 \;\; \mathbf{m}_2 \;\; \cdots \;\; \mathbf{m}_n]}_{\mathbf{M}} \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & 0 & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ \vdots & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}}_{\widetilde{\mathbf{A}}_c} = \mathbf{A}[\mathbf{m}_1 \;\; \mathbf{m}_2 \;\; \cdots \;\; \mathbf{m}_n]$$

$$\text{and } [\mathbf{m}_1 \;\; \mathbf{m}_2 \;\; \cdots \;\; \mathbf{m}_n] \underbrace{\begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\widetilde{\mathbf{B}}_c} = \mathbf{B}$$

The identification of $\mathbf{M}\widetilde{\mathbf{B}}_c = \mathbf{B}$ provides the last column of matrix $\mathbf{M}$: $\mathbf{m}_n = \mathbf{B}$.

The identification of the last column of $\mathbf{M}\widetilde{\mathbf{A}}_c = \mathbf{A}\mathbf{M}$ provides $\mathbf{m}_{n-1}$: $\mathbf{m}_{n-1} = (\mathbf{A} + a_{n-1}\mathbf{I}_n)\mathbf{B}$.

The identification of the column $n-1$ of $\mathbf{M}\widetilde{\mathbf{A}}_c = \mathbf{A}\mathbf{M}$ provides $\mathbf{m}_{n-2}$: $\mathbf{m}_{n-2} = (\mathbf{A}^2 + a_{n-1}\mathbf{A} + a_{n-2}\mathbf{I}_n)\mathbf{B}$.

$$\vdots$$

The identification of the column 2 of $\mathbf{M}\widetilde{\mathbf{A}}_c = \mathbf{A}\mathbf{M}$ provides $\mathbf{m}_1$: $\mathbf{m}_1 = (\mathbf{A}^{n-1} + a_{n-1}\mathbf{A}^{n-2} + \cdots + a_2\mathbf{A} + a_1\mathbf{I}_n)\mathbf{B}$.

Since all the colmuns $\mathbf{m}_i$ of matrix $\mathbf{M}$ are now defined, we have to check that the identification of the column 1 of $\mathbf{M}\widetilde{\mathbf{A}}_c = \mathbf{A}\mathbf{M}$ works. That is confirmed thanks to the **Cayley-Hamilton** theorem (see:

`CayleyHamilton`

). Indeed, we get:

$$\underbrace{(\mathbf{A}^n + a_{n_1}\mathbf{A}^{n-1} + \cdots + a_1\mathbf{A} + a_0\mathbf{I}_n)}_{=\mathbf{0}\ (\text{Cayley-Hamilton theorem})}\mathbf{B} = \mathbf{0} \; .$$

Finally, we have to check that $\mathbf{M}$ is invertible. Indeed, since all its columns $\mathbf{m}_i$ are linear combinations of $\mathbf{B}$, $\mathbf{AB}$, $\mathbf{A}^2\mathbf{B}$, $\cdots$ $\mathbf{A}^{n-1}\mathbf{B}$, then:

$$\mathbf{M} \text{ is invertible iff rank} \left(\underbrace{\begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}}_{\mathcal{C}} \right) = n \quad \bullet$$

Once $\widetilde{\mathbf{K}}$ and $\mathbf{M}$ are known, one can compute the state feedback gain $\mathbf{K} = \widetilde{\mathbf{K}}\mathbf{M}^{-1}$ to be applied on the initial **physical** state $\mathbf{x}$. Indeed:

$$u = -\widetilde{\mathbf{K}}\widetilde{\mathbf{x}} = -\underbrace{\widetilde{\mathbf{K}}\mathbf{M}^{-1}}_{\mathbf{K}}\mathbf{x} = -\mathbf{K}\mathbf{x} \; .$$

Illustration: for a given n-th order system $(\mathbf{A},\ \mathbf{B})$ and a given vector of n specifed eigenvalues $\mathbf{P} = [\lambda_1,\ \lambda_2, \cdots,\ \lambda_n]$, the computation of $\widetilde{\mathbf{K}}$, $\mathbf{M}$ and $\mathbf{K}$ are embedded in the MATLAB function `acker` (see also:

`https://en.wikipedia.org/wiki/Ackermann%27s_formula`

):

```
help acker
```

```
 ACKER  Pole placement gain selection using Ackermann's formula.

    K = ACKER(A,B,P)  calculates the feedback gain matrix K such that
    the single input system
            .
            x = Ax + Bu

    with a feedback law of  u = -Kx  has closed loop poles at the
    values specified in vector P, i.e.,  P = eig(A-B*K).

    Note: This algorithm uses Ackermann's formula.  This method
    is NOT numerically reliable and starts to break down rapidly
    for problems of order greater than 10, or for weakly controllable
    systems.  A warning message is printed if the nonzero closed-loop
    poles are greater than 10% from the desired locations specified
    in P.

    See also  PLACE.
```

Considering the previous example $G(s)A(s)$, one can choose to assign the 4 closed loop eigenvalues to:

- $-1 \pm j$: i.e. the low-frequency closed-loop dynamic corresponds to a stable 2-nd order with a damping ratio of $\sqrt{2}/2$ and a frequency of $\sqrt{2}\, rd/s$,
- the 2 roots of $s^2 + 14s + 100$: i.e. the actuator dynamics is unchanged.

```
P=[-1+sqrt(-1);-1-sqrt(-1);roots([1 14 100])];
K=acker(A,B,P);
damp(A-B*K)
% Closed-loop system with the 4 states and the control signal on the
% output:
G_CL=ss(A-B*K,B,[eye(4);-K],0);
% Simulation
figure
initial(G_CL,x0)
%
```

```
        Pole                Damping       Frequency       Time Constant
                                        (rad/TimeUnit)     (TimeUnit)

 -7.00e+00 + 7.14e+00i     7.00e-01       1.00e+01          1.43e-01
 -7.00e+00 - 7.14e+00i     7.00e-01       1.00e+01          1.43e-01
 -1.00e+00 + 1.00e+00i     7.07e-01       1.41e+00          1.00e+00
 -1.00e+00 - 1.00e+00i     7.07e-01       1.41e+00          1.00e+00
```