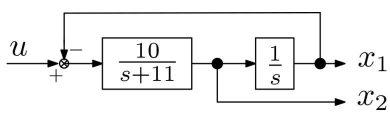
Dominant mode: illustration

Contents

1. A second order system:

Let us consider the system $G(s)$ described in the following Figure.



This system can be represented by the transfer functions:

- $\frac{X_1}{U}(s) = \frac{10}{s^2+11s+10} = \frac{10}{(s+1)(s+10)}$
- $\frac{X_2}{U}(s) = \frac{10s}{s^2+11s+10} = \frac{10s}{(s+1)(s+10)}$

or the state space representation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u \ \ (1)\,, \quad \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ \ (2)\,.$$

Its dynamics is characterized by 2 poles or eigenvalues:

- $\lambda_1 = -1\,(rd/s)$: the "slow" one,
- $\lambda_2 = -10\,(rd/s)$: the "fast" one.

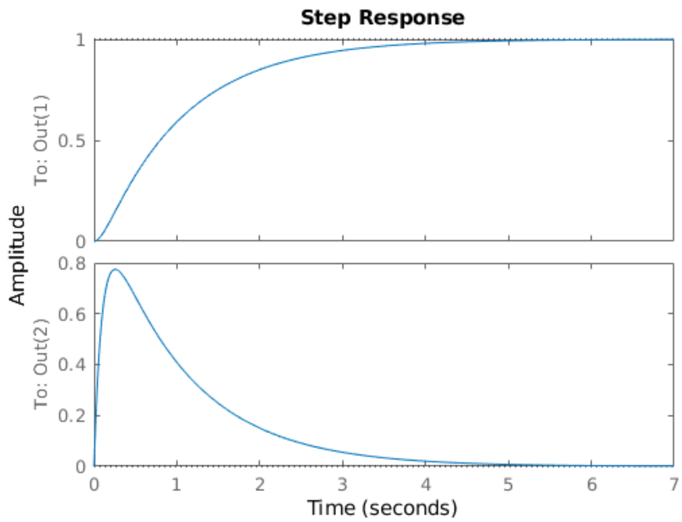
```
G=ss([0 1;-10 -11],[0;10],eye(2),zeros(2,1));
damp(G)
```

```
   Pole          Damping        Frequency       Time Constant
                              (rad/seconds)       (seconds)

 -1.00e+00      1.00e+00        1.00e+00          1.00e+00
 -1.00e+01      1.00e+00        1.00e+01          1.00e-01
```

Its step response:

```
figure
step(G)
```



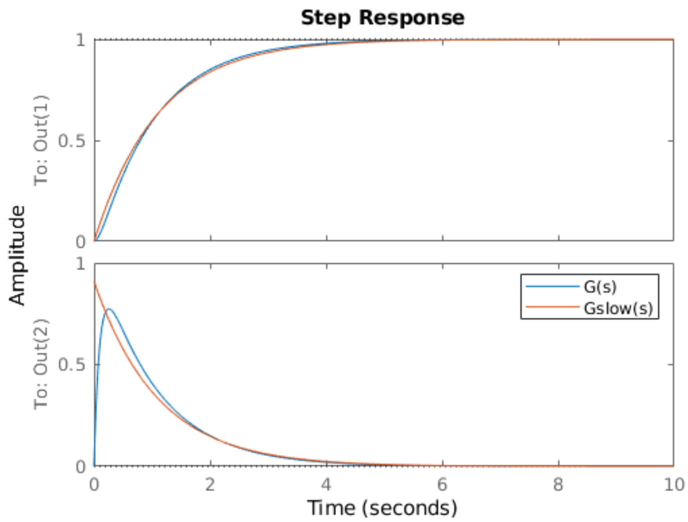
2. Long-term dominant mode:

Clearly the responses (it is more particularly true for $x_1(t)$) are dominated by the slow eigenvalue λ_1. Indeed, the long-term response in the time-domain corresponds to the low-frequency response in the frequency-domain, i.e. when $s = j\omega$ tends to 0. On the transfer functions, that means the terms in s^2 can be neglected w.r.t to terms in s or s^0:

- $\lim_{s\to 0} \frac{X_1}{U}(s) = \lim_{s\to 0} \frac{10}{11s+10}$.
- $\lim_{s\to 0} \frac{X_2}{U}(s) = \lim_{s\to 0} \frac{10s}{11s+10}$.

Thus, it is possible to find a first order system $G_{slow}(s) = \frac{\begin{bmatrix} 10 \\ 10s \end{bmatrix}}{11s+10}$ quite representative of the long-term response of the 2nd order system $G(s)$. Such an operation is called **a model reduction**.

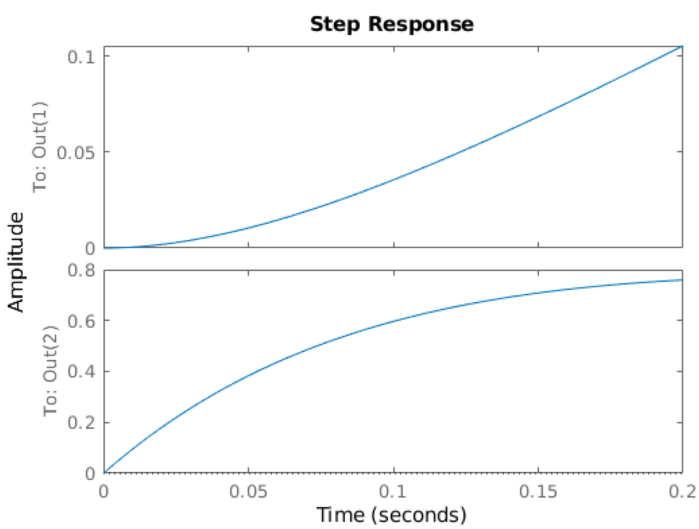
```
Gslow=tf({10;[10 0]},[11 10]);
hold on
step(Gslow)
legend('G(s)','Gslow(s)')
```



3. Short-term dominant mode:

The short-term step response ($t \in [0, \;\; 0.2\,s]$):

```
figure
step(G,0.2)
```

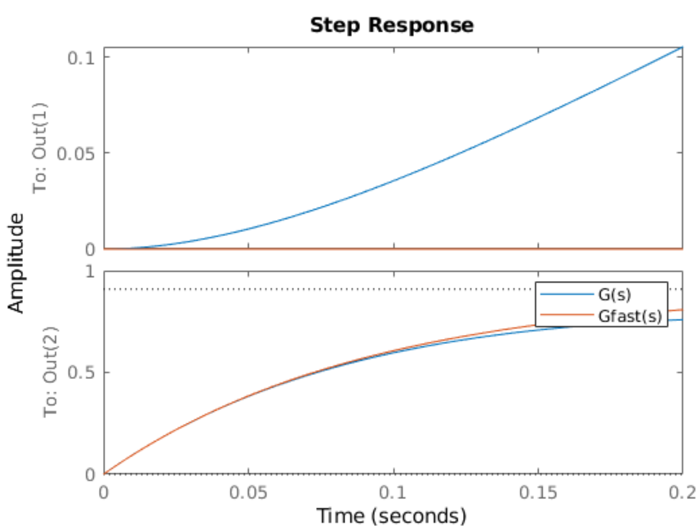


Clearly the response $x_2(t)$ is dominated by the fast eigenvalue λ_2. Indeed, the short-term response in the time-domain corresponds to the high-frequency response in the frequency-domain, i.e. when $s = j\omega$ tends to ∞. On the transfer functions, that means the terms in s^0 can be neglected w.r.t to terms in s or s^2:

- $\lim_{s\to\infty} \frac{X_1}{U}(s) = \lim_{s\to\infty} \frac{10}{s^2+11s} = 0$.
- $\lim_{s\to\infty} \frac{X_2}{U}(s) = \lim_{s\to\infty} \frac{10s}{s^2+11s} = \lim_{s\to\infty} \frac{10}{s+11}$.

Thus, it is possible to find a first order transfer quite representative of the short-term response of $x_2(t)$. For $x_1(t)$, it is not possible to find a first order transfer representative of its short-term behavior. We will choose 0 and thus: $G_{fast}(s) = \frac{\begin{bmatrix} 0 \\ 10 \end{bmatrix}}{s+11}$. Such an operation is also called a model reduction but it is completely different from the previous one.

```
Gfast=tf({0;10},[1 11]);
hold on
step(Gfast)
legend('G(s)','Gfast(s)')
```



4. Model simplification by state elimination.

The previous operations on transfer functions become tricky when we have have to cope with high order systems, with several inputs and outputs (ex: the longitudinal model of an aircraft). It is thus highly recommended to eliminate state variables directly in the state-space representation (see: `doc modred`). The function `modred` proposes 2 methods to eleminate state space variables:

- **'matchdc'** (i.e.: match DC gain, the **defauft method**): to be used to eliminate the "fast dynamics" in order to have a reduced model representative of the long-term behavior.

In our example x_2 is fast w.r.t. x_1, i.e.: x_2 reaches its steady state almost instaneoulsly w.r.t. to the settling time of x_1. So we will assume $\dot{x}_2 = 0$ (x_2 does not evolve any more). Then, in the second row of the state equation (1), we get: $x_2 = -\frac{10}{11}x_1 + \frac{10}{11}u$. By replacing x_2 in the first row of the state equation and in the output equation (2), the first order **reduced model** reads:

$$\dot{x}_1 = -\frac{10}{11}x_1 + \frac{10}{11}u, \quad \mathbf{y} = \begin{bmatrix} 1 \\ -\frac{10}{11} \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ \frac{10}{11} \end{bmatrix} u$$

```
Gslow=modred(G,2);
tf(Gslow)
```

```
ans =

  From input to output...
         0.9091
   1:  ----------
       s + 0.9091

        0.9091 s
   2:  ----------
       s + 0.9091

Continuous-time transfer function.
```

- **'truncate'**: to be used to eliminate the "slow dynamics" in order to have a reduced model representative of the short-term behavior.

In our example x_1 is slow w.r.t to x_2, i.e.: within the time constant of x_2, x_1 has not time enough to change from its equilibrium value. So we will assume $x_1 = 0$. Then considering the second row of the state equation and the output equation (2), the first order **reduced model** reads:

$$\dot{x}_2 = -11x_2 + 10u, \quad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2$$

```
Gfast=modred(G,1,'truncate');
tf(Gfast)
```

```
ans =

  From input to output...
   1:  0

         10
   2:  ------
       s + 11

Continuous-time transfer function.
```