Observability:

Contents

1. General result:

A linear continuous-time time-invariant system: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ is observable iff: from the observation of $\mathbf{y}([t_0,\ \ t_f])$ for a given final time t_f, it is possible possible to determine the initial value of the state $\mathbf{x}(t_0)$.

Let n be the system order ($\mathbf{x} \in \mathbb{R}^n$). The observability property depends only on matrices $\mathbf{A}$ and $\mathbf{C}$. In the sequel, we will consider the pair $(\mathbf{A}, \mathbf{C})$ instead of the 4 matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ of the system.

The pair $(\mathbf{A}, \mathbf{C})$ is **observable** iff:

$$\text{rank}(\mathcal{O}) = n \;\text{ with: } \mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}.$$

$\mathcal{O}$ is called the **observability matrix** .

2. Proof:

Let us consider the first $n-1$ time-derivative of the free ($\mathbf{u}(t) = 0,\ \forall\, t$) response of the output $\mathbf{y}(t)$:

$$\begin{aligned} \mathbf{y} &= \mathbf{C}\mathbf{x} \\ \dot{\mathbf{y}} &= \mathbf{C}\dot{\mathbf{x}} = \mathbf{C}\mathbf{A}\mathbf{x} \\ \ddot{\mathbf{y}} &= \mathbf{C}\ddot{\mathbf{x}} = \mathbf{C}\mathbf{A}^2\mathbf{x} \\ &\vdots \\ \frac{d^{n-1}\mathbf{y}}{dt^{n-1}} &= C\frac{d^{n-1}\mathbf{x}}{dt^{n-1}} = \mathbf{C}\mathbf{A}^{n-1}\mathbf{x} \end{aligned}$$

or:

$$\begin{bmatrix} \mathbf{y} \\ \dot{\mathbf{y}} \\ \ddot{\mathbf{y}} \\ \vdots \\ \frac{d^{n-1}\mathbf{y}}{dt^{n-1}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}}_{\mathcal{O}} \mathbf{x}\,.$$

In the (non-restrictive) single output case ($\mathbf{y}(t) \rightarrow y(t)$), $\mathcal{O}$ is a $n \times n$ matrix. Thus, at any time $t \in [t_0, \;\; t_f]$, on can determine $\mathbf{x}(t)$ from $\dot{y}$, $\ddot{y}$, $\cdots$, $\frac{d^{n-1}y}{dt^{n-1}}$ iff rank($\mathcal{O}$) $= n$:

$$\mathbf{x}(t) = \mathcal{O}^{-1} \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ \vdots \\ \frac{d^{n-1}y}{dt^{n-1}} \end{bmatrix} \quad \text{and} \quad \mathbf{x}(t_0) = \mathrm{e}^{\mathbf{A}(t_0-t)}\mathbf{x}(t) \quad \bullet$$

Remark: in this approach, it is assumed that one can derive $y(t)$ at any time t in $[t_0, \;\; t_f]$ using perfect non causal derivators (i.e.: knowing the whole trajectory $y([t_0, \;\; t_f])$). In a real-time implementation, we must keep in mind that such a perfect derivation is not **realizable**.

3. Duality:

The dual system G_d(s) of the primal system G_p(s) defined by the 4 state-space matrices **A**, **B**, **C**, **D** also noted:

$$G_p(\mathrm{s}) \equiv \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$$

is:

$$G_d(\mathrm{s}) \equiv \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]^T = \left[\begin{array}{c|c} \mathbf{A}^T & \mathbf{C}^T \\ \hline \mathbf{B}^T & \mathbf{D}^T \end{array}\right] .$$

The controllablity (resp. observability) conditions can be converted into observability (resp. controllabiity) conditions by changing the pair $(\mathbf{A}, \mathbf{B})$ by the pair $(\mathbf{A}^T, \mathbf{C}^T)$ (resp $(\mathbf{A}, \mathbf{C})$ by $(\mathbf{A}^T, \mathbf{B}^T)$).