Demo on the interpretation of the H_∞ norm of linear systems:

Let us consider a linear system $\mathbf{G}(s)$ with:

- $n = 6$ states,
- $m = 2$ inputs and
- $p = 3$outputs:

```
warning('off');
G=ss([-0.5 5 0 0 0 0;-5 -0.5 0 0 0 0;0 0 -8 0 0 0;...
      0 0 0 -8 0 0;0 0 0 0 -4.5 1.5; 0 0 0 0 -1.5 -4.5],...
     [0.5 -0.9;0.5 0.4;1.2 0.13;1.5 -2.33; -0.5 -0.3;2.2 1.8],...
     [-0.7 1.3 0.2 1.1 0.75 0.2;-0.4 -1.1 -1 -2.3 0.2 0.5;...
      -0.4 -0.5 0 0.9 0.4 0.5],[-0.9 0;1.2 -3.1;1.6 0.6]);
size(G)
```

```
State-space model with 3 outputs, 2 inputs, and 6 states.
```

By definition $\|\mathbf{G}(s)\|_\infty = \max_\omega \sigma_{\max}(\mathbf{G}(j\omega))$

```
norm(G,'inf')
```

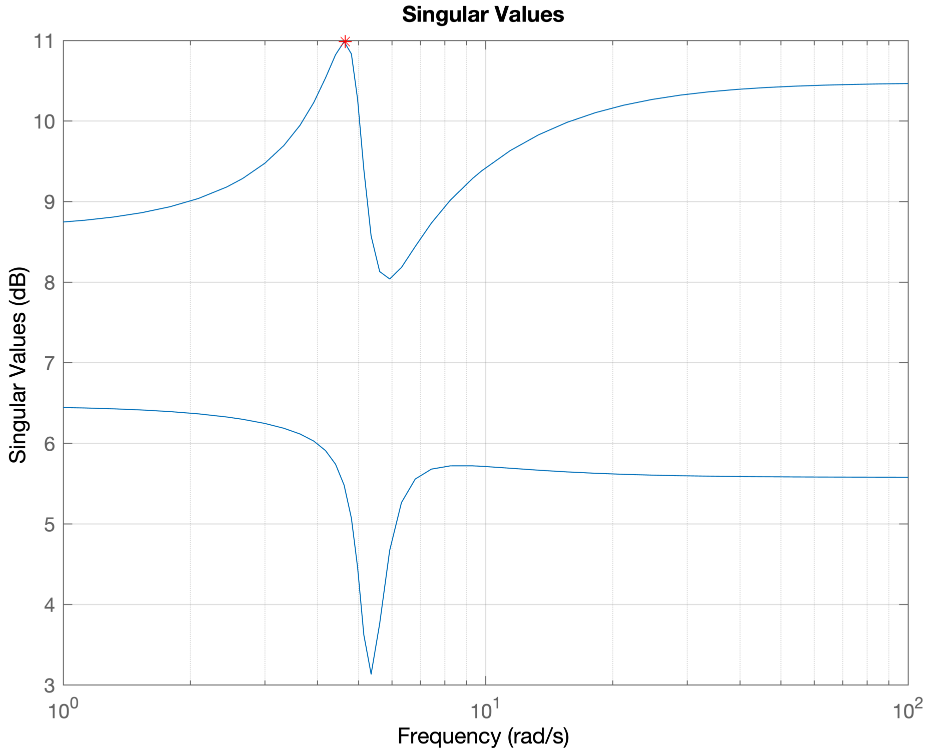
```
ans = 3.5431
```

$\|\mathbf{G}(s)\|_\infty$ is the peak of the frequency-domain response of the singular values of $\mathbf{G}(j\omega)$. The frequency $\omega_{wc}(rad/s)$ of the peak can be found thanks to the function **`hinfnorm`**:

```
[gain,w_wc]=hinfnorm(G)
```

```
gain = 3.5431
w_wc = 4.6517
```

```
figure
sigma(G);
grid
hold on
plot(w_wc,20*log10(gain),'r*')
```



The property: $\|\mathbf{G}(\mathrm{s})\|_\infty = \sup_{U(s)\in H_2} \frac{\|\mathbf{Y}(\mathrm{s})\|_2}{\|\mathbf{U}(\mathrm{s})\|_2}$ can be interpreted considering the steady state harmonic response of the system at the frequency ω_{wc}. Indeed the Singular Value Decomposition (SVD) of $\mathbf{G}(j\omega_{wc})$ reads:

$$\mathbf{G}(j\omega_{wc}) = [\mathbf{u}_1 \;\; \mathbf{u}_2 \;\; \mathbf{u}_3] \begin{bmatrix} \sigma_{\max}(\omega_{wc}) & 0 \\ 0 & \sigma_{\min}(\omega_{wc}) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^* \\ \mathbf{v}_2^* \end{bmatrix} .$$

(x^* is the transconjugated of x.). All $\mathbf{u}_i (i = 1, \cdots p)$ and $\mathbf{v}_j$ $(j = 1 \cdots m)$ are unitarty vectors. Thus:

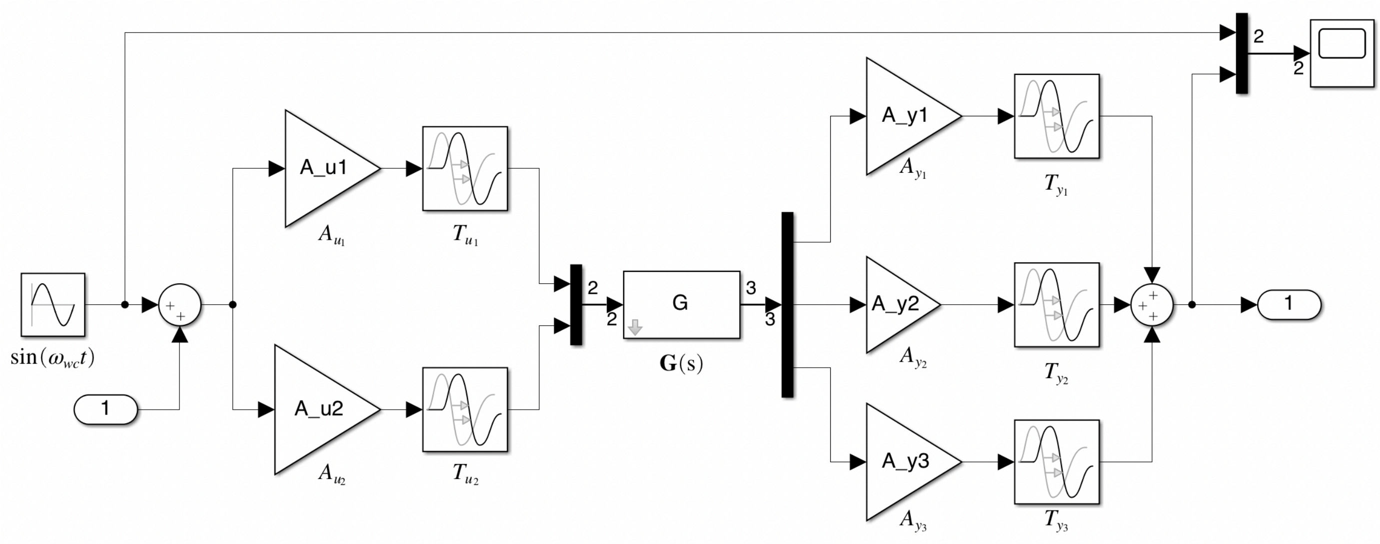
$\sigma_{\max}(\omega_{wc}) = \|\mathbf{G}(\mathrm{s})\|_\infty = \mathbf{u}_1^* \mathbf{G}(j\omega_{wc})\mathbf{v}_1$.

The complex components of $\mathbf{v}_1$(resp. $\mathbf{u}_1^*$) gives the combination of the harmonic inputs (resp. outputs) leading to the worst (or the highest) magnitude at the frequency ω_{wc}.

Each component of $\mathbf{u}_1^*$ or $\mathbf{v}_1$ can be expressed under the form $A\mathrm{e}^{j\phi}$ where A is the magnitude and ϕ is the phase or under the form $A\mathrm{e}^{-jT\omega_{wc}}$ where T is a delay providing the phase lag ϕ at the frequency ω_{wc}.:

- $T = -\phi/\omega_{wc}$, if $\phi \in [0, -\pi]$,
- $T = -(\phi - 2\pi)/\omega_{wc}$, if $\phi \in [0, \pi]$.

Thus, from the 3×2 MIMO system $\mathbf{G}(s)$, it is possible to build a SISO real system whose the highest magntitude frequency response is at ω_{wc} with a magnitude of $\|\mathbf{G}(s)\|_\infty$. This system is described by the following block-diagram



with:

$$\mathbf{v}_1 = \begin{bmatrix} A_{u_1} e^{-jT_{u_1}\omega_{wc}} \\ A_{u_2} e^{-jT_{u_2}\omega_{wc}} \end{bmatrix} \text{ and } \mathbf{u}_1^* = \begin{bmatrix} A_{y_1} e^{-jT_{y_1}\omega_{wc}} & A_{y_2} e^{-jT_{y_2}\omega_{wc}} & A_{y_3} e^{-jT_{y_3}\omega_{wc}} \end{bmatrix}.$$

The following sequence illustrated the frequency-response at ω_{wc}.

```
G_M=freqresp(G,w_wc);
[U,S,V]=svd(G_M);
U(:,1)'*G_M*V(:,1)
```

```
ans = 3.5431 - 0.0000i
```

This results illustrates that: $\sigma_{\max}(\omega_{wc}) = \|\mathbf{G}(s)\|_\infty = \mathbf{u}_1^*\mathbf{G}(j\omega_{wc})\mathbf{v}_1$.

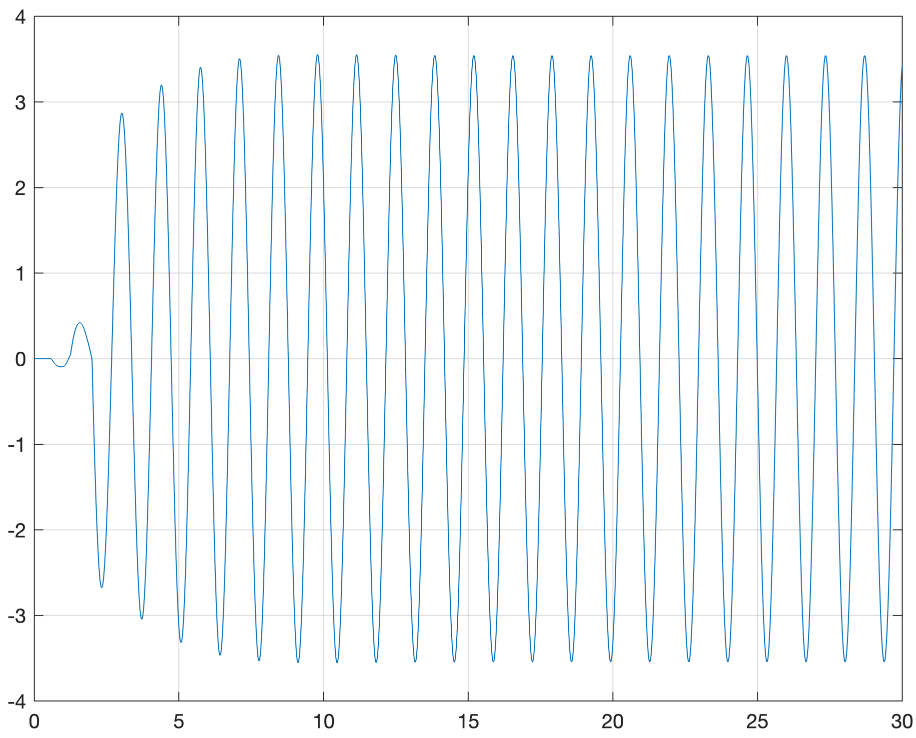
```
% Worst magnitude and phase distribution for harmonic inputs at w_wc:
% (with the constraint sum(A_ui^2)=1)
A_u1=abs(V(1,1));
phi_u1=angle(V(1,1));
if phi_u1>0,phi_u1=phi_u1-2*pi;end
A_u2=abs(V(2,1));
phi_u2=angle(V(2,1));
if phi_u2>0,phi_u2=phi_u2-2*pi;end
% phase equivalent delay at the frequency w_wc:
T_u1=-phi_u1/w_wc;
T_u2=-phi_u2/w_wc;

% Worst magnitude and phase combination of the outputs:
% (with the constraint sum(A_yi^2)=1)
```

```
A_y1=abs(U(1,1)');
phi_y1=angle(U(1,1)');
if phi_y1>0,phi_y1=phi_y1-2*pi;end
A_y2=abs(U(2,1)');
phi_y2=angle(U(2,1)');
if phi_y2>0,phi_y2=phi_y2-2*pi;end
A_y3=abs(U(3,1)');
phi_y3=angle(U(3,1)');
if phi_y3>0,phi_y3=phi_y3-2*pi;end
% phase equivalent delay at the frequency w_wc:
T_y1=-phi_y1/w_wc;
T_y2=-phi_y2/w_wc;
T_y3=-phi_y3/w_wc;
```

Simulation: the following SIMULINK system is a SISO real system with the same H_∞ norm than $\mathbf{G}(s)$ (up to the approximation of the delay).

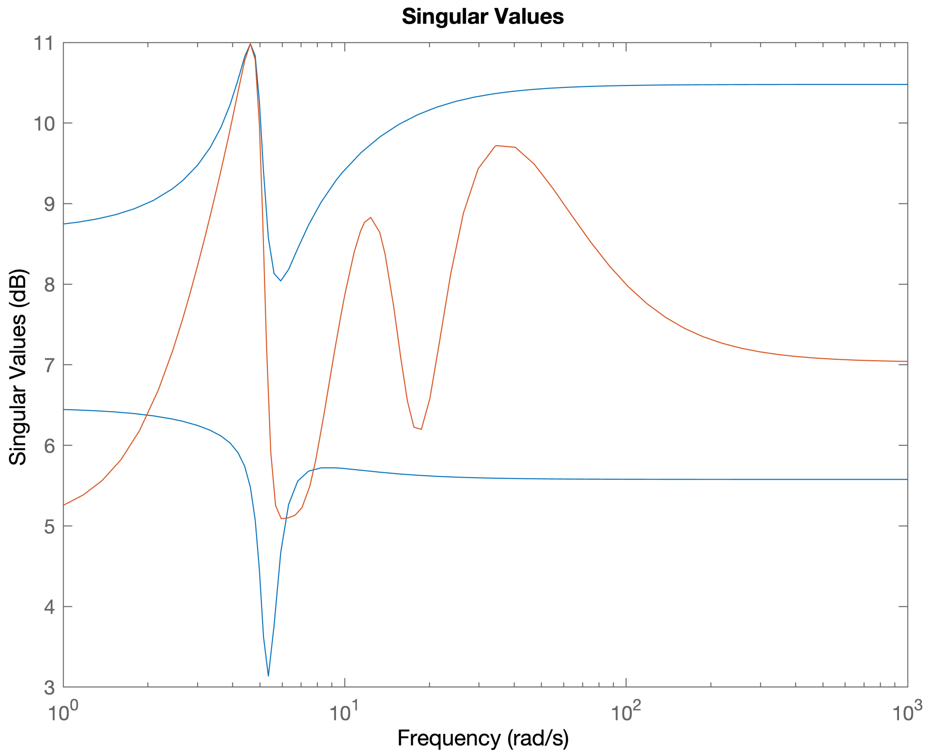
```
simul_wc_harmonic
SimOut=sim('simul_wc_harmonic');
figure
plot(SimOut.yout{1}.Values.time,SimOut.yout{1}.Values.Data)
grid
```



The magnitude of the oscillations (in steady state) is equal to $\|\mathbf{G}(s)\|_\infty$.

```
[a,b,c,d]=linmod('simul_wc_harmonic');
figure
```

```
sigma(G,ss(a,b,c,d));
```



The SISO SIMULINK syqstem and $\mathbf{G}(s)$ have the same H_∞ norm.

Thus, for a MIMO system, $\|\mathbf{G}(s)\|_\infty$ gives the magnitude in steady state of the response of the "worst" complex combination of the outputs to an harmonic signal at the critical frequency ω_{wc} with the worst complex distribution on the inputs.