Notes on Laminates Optimization

Joseph Morlier / Dimitri Bettebghor

January 2010

Summary

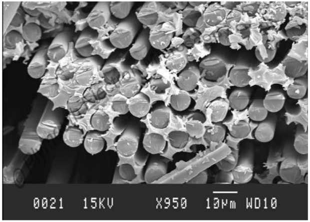
1. Overview of Composites
2. Ply mechanics
3. Classical Laminates Theory
4. Laminates optimization: laminated Parameters

Definition

Composite materials (Latin: componere) "'A macroscopic combination of two or more distinct materials into one with the intent of surpressing undesirable properties of the constituent materials in favour of desirable properties"'

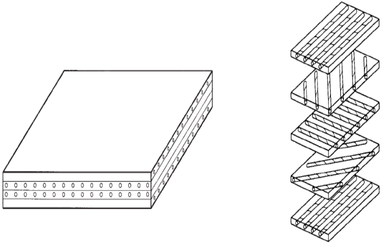
At Macro Level

Isotropic
"'The same properties in all directions"' (Metals, plastics, concrete, etc)
Anisotropic
"'Different properties in different directions"' (Wood, reinforced concrete, fibre composites, bone, and almost all natural building materials)
An example would be the dependence of Young's modulus on the direction of load



Laminates composites

- Combination of two or more constituent materials on a macroscopic examination to produce a new material with enhanced properties: Fibers (carbon, glass, Kevlar) and matrix (Epoxy resin)

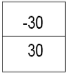


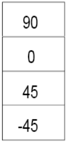
- Strength and stiffness are proportional to the amount of fibers in the matrix (Fiber volume fraction)
- The reinforcing fibers provide the useful engineering properties e.g., strength and stiffness); whereas the matrix serves to protect and stabilize the fibers while transferring loads among the fibers predominantly through shear.

Stacking Sequence

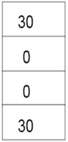
Description of Laminate

- The layers are numbered starting at the bottom and the angles are given from bottom up (e.g., [30°/-30°], [-45°/45°/0°/90°])

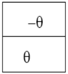




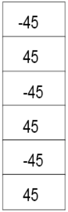
- Symmetric laminate: [30°/0°/0°/30°] = $[30°/0°]_S$



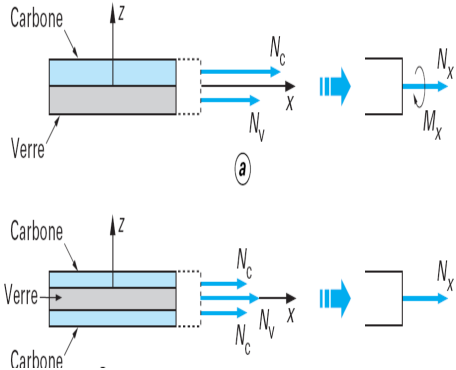
- Angle-ply combination: [θ/–θ] = [+/-θ]



- Repeating pattern: [45/-45/45/-45/45/-45] = $[(+/-45)_3]_T$

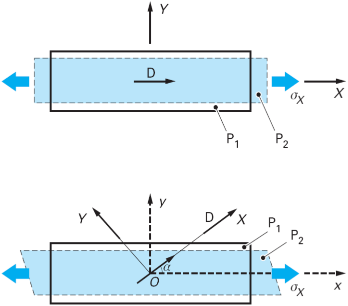


Mirror symetry?



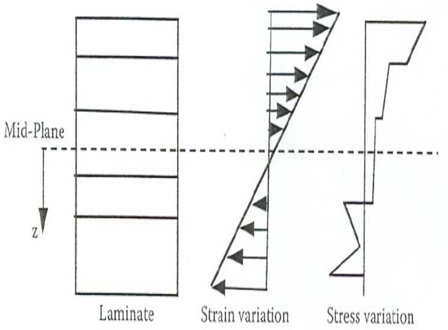
Coupling terms

- When the local coordinate changes : some terms of Compliance matrix are different of Zero (Distortion of right angle)



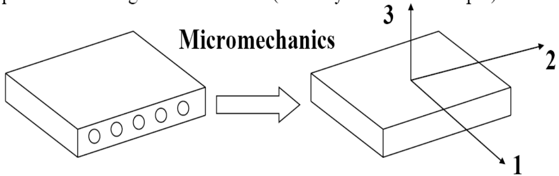
- For anistropic material, Stress-Strain Equations depend on Coordinate.

Stress and Strain in a laminate

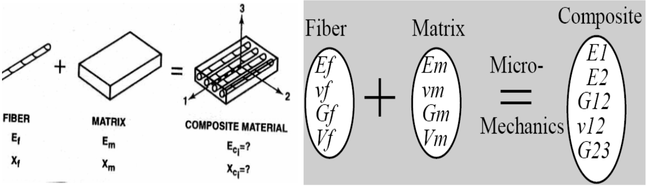


Equivalent material

- To represent a Heterogeneous (Inhomogeneous) material with an equivalent Homogeneous material (but may be in Anisotropic)



- Prediction of Stiffness Properties for Equivalent Homogeneous Material (***E*1, *E*2, *G*12, *G*23, *v*12**) along the material (principal) coordinates:



Simple rules: Mixture Law

$$E_L = E_1 = E_f V_f + E_m V_m$$

$$\frac{1}{E_T} = \frac{1}{E_2} = \frac{V_f}{E_f} + \frac{V_m}{E_m}$$

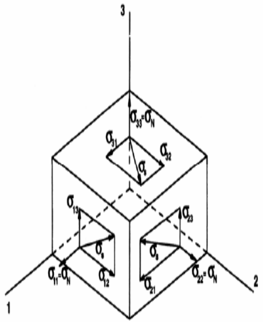
$$\frac{1}{G_{12}} = \frac{1}{G_{LT}} = \frac{V_f}{G_f} + \frac{V_m}{G_m}$$

$$\nu_{12} = \nu_{LT} = \nu_f V_f + \nu_m V_m$$

Elasticity Review: stress tensor

– Stress tensor: introduce to have a quantity that completely characterizes the state of stress at each point, valid for any cutting plane (Fig. 5.4).

$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$



stress tensor: Matrix notation

– Stress tensor is symmetric (Fig. 5.4):

$$\sigma_{ij} = \sigma_{ji}$$

Stress tensor in Matrix form:

$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

Contracted Notation

σ_{ij} *Tensorial*	σ_α *Contracted*
σ_{11}	σ_1
σ_{22}	σ_2
σ_{33}	σ_3
σ_{23}	σ_4
σ_{13}	σ_5
σ_{12}	σ_6

$i=j, \alpha = i$
$i \neq j, \alpha = 9 - i - j$

Plane stress assumption

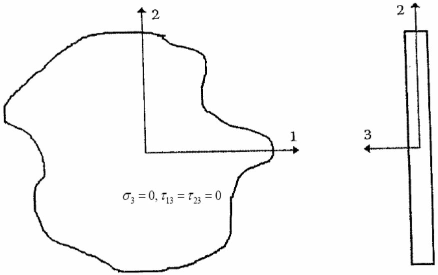


FIGURE 2.17
Plane stress conditions for a thin plate.

- Upper and lower surfaces are free from external loads
- $\sigma_3 = 0, \tau_{13} = \tau_{23} = 0$

  $\varepsilon_3 = S_{13}\sigma_1 + S_{23}\sigma_2, \gamma_{23} = \gamma_{31} = 0$

strain tensor: Matrix notation

- Engineering strain: change of length over original length

$$\varepsilon_x = \lim_{\Delta L \to 0} \frac{\Delta L}{L} = \frac{du}{dx}$$

- In 3-D, the displacement vs. strain are

Normal Strain	Shear Strain (Eng.)
$\varepsilon_{xx} = \frac{\partial u}{\partial x}$	$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$
$\varepsilon_{yy} = \frac{\partial v}{\partial x}$	$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$
$\varepsilon_{zz} = \frac{\partial w}{\partial x}$	$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$

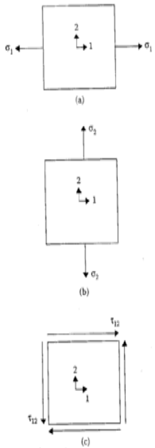
$$[\varepsilon] = \begin{bmatrix} \varepsilon_{11} & \gamma_{12}/2 & \gamma_{13}/2 \\ \gamma_{12}/2 & \varepsilon_{22} & \gamma_{23}/2 \\ \gamma_{13}/2 & \gamma_{23}/2 & \varepsilon_{33} \end{bmatrix}$$

Contracted Notation

Tensorial	*Contracted*
ε_{11}	ε_1
ε_{22}	ε_2
ε_{33}	ε_3
γ_{23}	γ_4
γ_{13}	γ_5
γ_{12}	γ_6

Stress-Strain Equations in Material Coordinate

In material coordinate, we can calculate: **any load cases (in plane) is a combination of these 3 cases**



- (a) Pure tensile load in direction 1
- (b) Pure tensile load in direction 2
- (c) Pure shear stress in the plane of 1-2

Normal strain in coupon along O1 (Material Coordinate

When loading is normal to face 1 (normal stress σ_1) , the structure elongates along O1 (ϵ_1) and shortens along O2 (ϵ_2).

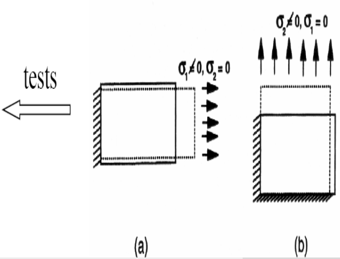
For composites with thin-walled structures, we can assume stress in the thickness direction is zero – plane stress:

$$\sigma_3 = 0$$

Compliance Coefficients (Fig. 5.8)

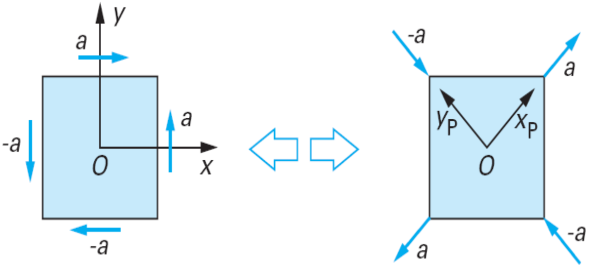
$$\varepsilon_1 = \frac{1}{E_1}\sigma_1 - \frac{\nu_{21}}{E_2}\sigma_2$$

$$\varepsilon_2 = -\frac{\nu_{12}}{E_1}\sigma_1 + \frac{1}{E_2}\sigma_2$$



Shear Loading

Under pure shearing force τ_{12} ,no ϵ_1 and ϵ_2 appears.
G_{12} (shear modulus) is defined by : $\tau_{12} = 2G_{12}\epsilon_{12}$



Stiffness Equations in Material Coordinate

any load cases (in plane) creating σ_1 , σ_2 , τ_{12} is a combination of these elementary cases:

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ 2\epsilon_{12} \end{pmatrix} = \begin{pmatrix} 1/E_1 & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{pmatrix}$$

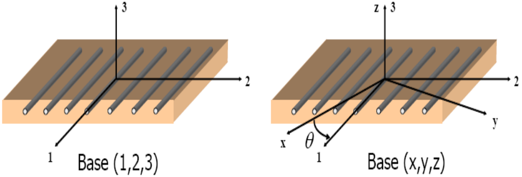
equivalent to (**Hooke**'s Law): $[\epsilon]$=[S]$[\sigma]$,
where [S] is the Compliance Matrix (9 terms)
Multiplying by S^{-1}, : $[\sigma]$=[Q"]$[\epsilon]$
we can obtain: $[Q'']$ Reduced Stiffness Matrix in the orthotropic coordinate:

$$[Q''] = \begin{pmatrix} \beta E_1 & \beta\nu_{12}E_2 & 0 \\ \beta\nu_{12}E_2 & \beta E_2 & 0 \\ 0 & 0 & G_{12} \end{pmatrix}$$

avec $\beta = \frac{1}{1-\nu_{12}\nu_{21}}$

Hooke's Law for a 2D Angle Lamina

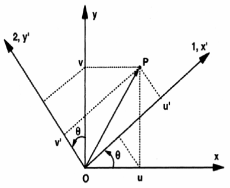
From local to global coordinate ?



$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} \qquad \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} \qquad \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Coordinate Transformations



$$u' = u\cos\theta + v\sin\theta$$
$$v' = -u\sin\theta + v\cos\theta$$
$(m = \cos\theta,\ n = \sin\theta)$

$$\Rightarrow \begin{Bmatrix} u' \\ v' \end{Bmatrix} = \begin{bmatrix} m & n \\ -n & m \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} m & -n \\ n & m \end{bmatrix} \begin{Bmatrix} u' \\ v' \end{Bmatrix}$$

The strains in material coordinates:

$$\varepsilon_1 = \frac{\partial u'}{\partial x'}$$
$$\varepsilon_2 = \frac{\partial v'}{\partial y'}$$
$$\gamma_6 = \gamma_{12} = \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}$$
$$\gamma_5 = \gamma_{13} = \frac{\partial u'}{\partial z'} + \frac{\partial w'}{\partial x'}$$
$$\gamma_4 = \gamma_{23} = \frac{\partial v'}{\partial z'} + \frac{\partial w'}{\partial y'}$$

In globle coordinates:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x}$$

and

$$\frac{\partial u}{\partial x'} = \frac{\partial u'}{\partial x'}m - \frac{\partial v'}{\partial x'}n$$
$$\frac{\partial u}{\partial y'} = \frac{\partial u'}{\partial y'}m - \frac{\partial v'}{\partial y'}n$$

where: $\frac{\partial x'}{\partial x} = m, \quad \frac{\partial y'}{\partial y} = -n$

thus: $$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u'}{\partial x'}m^2 - \frac{\partial v'}{\partial x'}mn - \frac{\partial u'}{\partial y'}mn + \frac{\partial v'}{\partial y'}n^2$$

$$\Rightarrow \varepsilon_x = m^2\varepsilon_1 - 2mn\frac{\gamma_6}{2} + n^2\varepsilon_2$$

Similarly for ε_y and ε_{xy} $(1/2\gamma_{xy})$

Coordinate Transformations

- Constitutive Equation in ***Material Coordinate*** is established;
- What is Constitutive Equation in ***Global Coordinate*** (Fig. 5.1)?

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} & ? & \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} \qquad \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} & ? & \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

- Transformation matrix $[T]$: the transformation between ***material*** and ***global*** coordinate systems.

$$[T] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

- Stress or strain:
  - Material = $[T]$ x Global
  - Global = $[T]^{-1}$ x Material

$$[T]^{-1} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & 2\sin\theta\cos\theta \\ \sin\theta\cos\theta & -\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

Stress Transformation

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \left[T_\sigma\right] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta-\sin^2\theta \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \left[T_\sigma\right]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & 2\sin\theta\cos\theta \\ \sin\theta\cos\theta & -\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

Strain Transformation

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \left[T_\varepsilon\right] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & 2\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \left[T_\varepsilon\right]^{-1} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & \sin\theta\cos\theta \\ 2\sin\theta\cos\theta & -2\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}$$

Stress-Strain relationship in Global Coordinate

base (1,2,3) :

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = [Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}$$

Relations transformation:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = [T_\sigma] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = [T_\varepsilon] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$[T_\sigma] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = [Q][T_\varepsilon] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Stress-Strain relationship in Global Coordinate

Multiplying by $[T]^{-1}$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \boxed{\left[T_\sigma^{-1}\right][Q]\left[T_\varepsilon\right]} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\left[\bar{Q}\right]$$

$$\left[\bar{S}\right] = \left[\bar{Q}\right]^{-1}$$

Stiffness and Compliance Matrix in Global Coordinate

$$
\begin{cases}
\bar{Q}_{11} = Q_{11} c^4 + 2(Q_{12} + 2Q_{66}) s^2 c^2 + Q_{22} s^4 \\
\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66}) s^2 c^2 + Q_{12}(s^4 + c^4) \\
\bar{Q}_{22} = Q_{11} s^4 + 2(Q_{12} + 2Q_{66}) s^2 c^2 + Q_{22} c^4 \\
\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66}) s c^3 + (Q_{12} - Q_{22} + 2Q_{66}) s^3 c \\
\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66}) s^3 c + (Q_{12} - Q_{22} + 2Q_{66}) s c^3 \\
\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) s^2 c^2 + Q_{66}(s^4 + c^4)
\end{cases}
$$

$$c = \cos\theta$$

$$s = \sin\theta$$

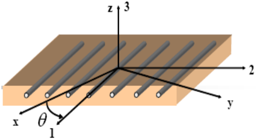
$$
\begin{cases}
\bar{S}_{11} = S_{11} c^4 + (2S_{12} + S_{66}) s^2 c^2 + S_{22} s^4 \\
\bar{S}_{12} = S_{12}(s^4 + c^4) + (S_{11} + S_{22} - S_{66}) s^2 c^2 \\
\bar{S}_{22} = S_{11} s^4 + (2S_{12} + S_{66}) s^2 c^2 + S_{22} c^4 \\
\bar{S}_{66} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66}) s^2 c^2 + S_{66}(s^4 c^4) \\
\bar{S}_{16} = (2S_{11} + 2S_{12} - S_{66}) s c^3 - (2S_{22} - 2S_{12} - S_{66}) s^3 c \\
\bar{S}_{26} = (2S_{11} + 2S_{12} - S_{66}) s^3 c - (2S_{22} - 2S_{12} - S_{66}) s c^3
\end{cases}
$$

Exemple 1

Carbon /époxy

$$\begin{cases} E_1 = 134000\ MPa \\ E_2 = 7000\ MPa \\ G_{12} = 4200\ MPa \\ \nu_{12} = 0{,}25 \end{cases}$$

$\theta = 45°$



$[\overline{Q}_{45}] = ?$

$[\overline{S}_{45}] = ?$

$E_{45} = ?$

Q45 =

1.0e+004 *

4.0443	3.2043	3.1854
3.2043	4.0443	3.1854
3.1854	3.1854	3.4488

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{16} \\ \overline{S}_{12} & \overline{S}_{22} & \overline{S}_{26} \\ \overline{S}_{16} & \overline{S}_{26} & \overline{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ 0 \\ 0 \end{Bmatrix}$$

$$\varepsilon_x = \overline{S}_{11}\sigma_x = \frac{1}{E_x}\sigma_x$$

S45 = inv(Q45)

1.0e-003 *

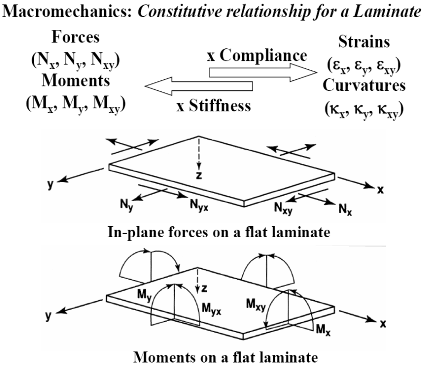
0.0962	-0.0229	-0.0677
-0.0229	0.0962	-0.0677
-0.0677	-0.0677	0.1541

$$E_{45} = \frac{1}{\overline{S}_{11}}$$

$$\frac{1}{E_{45}} = \frac{1}{4}\left(\frac{1}{E_1} + \frac{1}{E_2} - 2\frac{\nu_{12}}{E_1} + \frac{1}{G_{12}}\right)$$

E_{45}=10398 MPa

Macromechanics



Nx = normal force resultant in the x direction (per unit length) Nxy = shear force resultant (per unit length) Mx = bending moment resultant in the yz plane (per unit length) Mxy = twisting moment resultant (per unit length)

Displacement field

- Assumptions (Figure 6.2)
  - Line AD remains straight (shear strains γ_{xz} and γ_{yz} are constant through the thickness; accurate for thin laminates);
  - The length of Line AD is constant ($\varepsilon_{zz} = 0$).

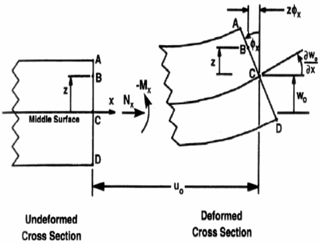


Fig. 6.2 Geometry of deformation in the x-z plane

- Displacement field:

$$u(x, y, z) = u_o(x, y) - z\phi_x(x, y)$$

$$v(x, y, z) = v_o(x, y) - z\phi_y(x, y)$$

$$w(x, y, z) = w_o(x, y)$$

where: u_o, v_o, and w_o are at the middle surface of the laminate,

and ϕ_x and ϕ_y are rotations around x and y axis, respectively.

Strain in the laminate

- **Strains: Classical Lamination Plate Theory (CLPT)**
  - *for a thin plate*

(Classical plate theory)

$$\gamma_{yz}(x,y,z) = -\phi_y + \frac{\partial w_0}{\partial y} = 0 \rightarrow \phi_y = \frac{\partial w_0}{\partial y}$$

$$\gamma_{xz}(x,y,z) = -\phi_x + \frac{\partial w_0}{\partial x} = 0 \rightarrow \phi_x = \frac{\partial w_0}{\partial x}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^{\,o} = \dfrac{\partial u_0}{\partial x} \\ \varepsilon_y^{\,o} = \dfrac{\partial v_0}{\partial y} \\ \gamma_{xy}^{\,o} = \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} + z \begin{Bmatrix} \kappa_x = -\dfrac{\partial \phi_x}{\partial x} = -\dfrac{\partial^2 w_0}{\partial x^2} \\ \kappa_y = -\dfrac{\partial \phi_y}{\partial y} = -\dfrac{\partial^2 w_0}{\partial y^2} \\ \kappa_{xy} = -\left(\dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x}\right) = -2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}$$

Strain in the laminate for each layer

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z_k \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}.$$

$\varepsilon_x^0, \varepsilon_y^0 =$ Midplane normal strains in the laminate

$\gamma_{xy}^0 =$ Midplane shear strain in the laminate

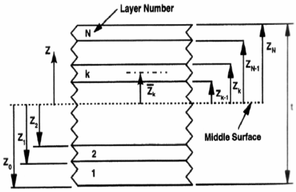
$k_x, k_y =$ Bending curvatures in the laminate

$k_{xy} =$ Twisting curvature in the laminate

$z =$ Distance from the midplane in the thickness direction

Plate resultant forces

- Stress Resultants (Figure 6.5):



$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-t/2}^{t/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} dz = \sum_{k=1}^{N} \int_{zk-1}^{zk} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}^k dz = \sum_{k=1}^{N} \int_{zk-1}^{zk} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}^k dz$$

$$\begin{Bmatrix} V_y \\ V_x \end{Bmatrix} = \int_{-t/2}^{t/2} \begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} dz = \sum_{k=1}^{N} \int_{zk-1}^{zk} \begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix}^k dz = \sum_{k=1}^{N} \int_{zk-1}^{zk} \begin{bmatrix} \bar{Q}^*_{44} & \bar{Q}^*_{45} \\ \bar{Q}^*_{45} & \bar{Q}^*_{55} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix}^k dz$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-t/2}^{t/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} z dz = \sum_{k=1}^{N} \int_{zk-1}^{zk} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}^k z dz = \sum_{k=1}^{N} \int_{zk-1}^{zk} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}^k z dz$$

Constitutive equation for laminated plate

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{zk-1}^{zk} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}^k dz = \sum_{k=1}^{N} \int_{zk-1}^{zk} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^k \left\{ \begin{Bmatrix} \varepsilon^0_y \\ \varepsilon^0_y \\ \varepsilon^0_{xy} \end{Bmatrix}^k + z \begin{Bmatrix} \kappa^0_y \\ \kappa^0_y \\ \kappa^0_{xy} \end{Bmatrix}^k \right\} dz$$

$$\begin{Bmatrix} V_y \\ V_x \end{Bmatrix} = \sum_{k=1}^{N} \int_{zk-1}^{zk} \begin{bmatrix} \bar{Q}^*_{44} & \bar{Q}^*_{45} \\ \bar{Q}^*_{45} & \bar{Q}^*_{55} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix}^k dz$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{zk-1}^{zk} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}^k zdz = \sum_{k=1}^{N} \int_{zk-1}^{zk} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^k \left\{ \begin{Bmatrix} \varepsilon^0_y \\ \varepsilon^0_y \\ \varepsilon^0_{xy} \end{Bmatrix}^k + z \begin{Bmatrix} \kappa^0_y \\ \kappa^0_y \\ \kappa^0_{xy} \end{Bmatrix}^k \right\} zdz$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} \sum_{k=1}^{N} (\bar{Q}_{11})_k \int_{z_{k-1}}^{z_k} dz & \sum_{k=1}^{N} (\bar{Q}_{12})_k \int_{z_{k-1}}^{z_k} dz & \sum_{k=1}^{N} (\bar{Q}_{16})_k \int_{z_{k-1}}^{z_k} dz \\ \sum_{k=1}^{N} (\bar{Q}_{12})_k \int_{z_{k-1}}^{z_k} dz & \sum_{k=1}^{N} (\bar{Q}_{22})_k \int_{z_{k-1}}^{z_k} dz & \sum_{k=1}^{N} (\bar{Q}_{26})_k \int_{z_{k-1}}^{z_k} dz \\ \sum_{k=1}^{N} (\bar{Q}_{16})_k \int_{z_{k-1}}^{z_k} dz & \sum_{k=1}^{N} (\bar{Q}_{26})_k \int_{z_{k-1}}^{z_k} dz & \sum_{k=1}^{N} (\bar{Q}_{66})_k \int_{z_{k-1}}^{z_k} dz \end{bmatrix}^k \begin{Bmatrix} \varepsilon^0_y \\ \varepsilon^0_y \\ \varepsilon^0_{xy} \end{Bmatrix}^k + \begin{bmatrix} \sum_{k=1}^{N} (\bar{Q}_{11})_k \int_{z_{k-1}}^{z_k} zdz & \sum_{k=1}^{N} (\bar{Q}_{12})_k \int_{z_{k-1}}^{z_k} zdz & \sum_{k=1}^{N} (\bar{Q}_{16})_k \int_{z_{k-1}}^{z_k} zdz \\ \sum_{k=1}^{N} (\bar{Q}_{12})_k \int_{z_{k-1}}^{z_k} zdz & \sum_{k=1}^{N} (\bar{Q}_{22})_k \int_{z_{k-1}}^{z_k} zdz & \sum_{k=1}^{N} (\bar{Q}_{26})_k \int_{z_{k-1}}^{z_k} zdz \\ \sum_{k=1}^{N} (\bar{Q}_{16})_k \int_{z_{k-1}}^{z_k} zdz & \sum_{k=1}^{N} (\bar{Q}_{26})_k \int_{z_{k-1}}^{z_k} zdz & \sum_{k=1}^{N} (\bar{Q}_{66})_k \int_{z_{k-1}}^{z_k} zdz \end{bmatrix}^k \begin{Bmatrix} \kappa^0_y \\ \kappa^0_y \\ \kappa^0_{xy} \end{Bmatrix}^k$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} \sum_{k=1}^{N} (\bar{Q}_{11})_k \int_{z_{k-1}}^{z_k} zdz & \sum_{k=1}^{N} (\bar{Q}_{12})_k \int_{z_{k-1}}^{z_k} zdz & \sum_{k=1}^{N} (\bar{Q}_{16})_k \int_{z_{k-1}}^{z_k} zdz \\ \sum_{k=1}^{N} (\bar{Q}_{12})_k \int_{z_{k-1}}^{z_k} zdz & \sum_{k=1}^{N} (\bar{Q}_{22})_k \int_{z_{k-1}}^{z_k} zdz & \sum_{k=1}^{N} (\bar{Q}_{26})_k \int_{z_{k-1}}^{z_k} zdz \\ \sum_{k=1}^{N} (\bar{Q}_{16})_k \int_{z_{k-1}}^{z_k} zdz & \sum_{k=1}^{N} (\bar{Q}_{26})_k \int_{z_{k-1}}^{z_k} zdz & \sum_{k=1}^{N} (\bar{Q}_{66})_k \int_{z_{k-1}}^{z_k} zdz \end{bmatrix}^k \begin{Bmatrix} \varepsilon^0_y \\ \varepsilon^0_y \\ \varepsilon^0_{xy} \end{Bmatrix}^k + \begin{bmatrix} \sum_{k=1}^{N} (\bar{Q}_{11})_k \int_{z_{k-1}}^{z_k} z^2dz & \sum_{k=1}^{N} (\bar{Q}_{12})_k \int_{z_{k-1}}^{z_k} z^2dz & \sum_{k=1}^{N} (\bar{Q}_{16})_k \int_{z_{k-1}}^{z_k} z^2dz \\ \sum_{k=1}^{N} (\bar{Q}_{12})_k \int_{z_{k-1}}^{z_k} z^2dz & \sum_{k=1}^{N} (\bar{Q}_{22})_k \int_{z_{k-1}}^{z_k} z^2dz & \sum_{k=1}^{N} (\bar{Q}_{26})_k \int_{z_{k-1}}^{z_k} z^2dz \\ \sum_{k=1}^{N} (\bar{Q}_{16})_k \int_{z_{k-1}}^{z_k} z^2dz & \sum_{k=1}^{N} (\bar{Q}_{26})_k \int_{z_{k-1}}^{z_k} z^2dz & \sum_{k=1}^{N} (\bar{Q}_{66})_k \int_{z_{k-1}}^{z_k} z^2dz \end{bmatrix}^k \begin{Bmatrix} \kappa^0_y \\ \kappa^0_y \\ \kappa^0_{xy} \end{Bmatrix}$$

Laminate stiffness matrix

- Constitutive equations for Laminate (Stiffness) (Figure 6.6):

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} \sum_{k=1}^{N} (\bar{Q}_{11})_k (z_k - z_{k-1}) & \sum_{k=1}^{N} (\bar{Q}_{12})_k (z_k - z_{k-1}) & \sum_{k=1}^{N} (\bar{Q}_{16})_k (z_k - z_{k-1}) \\ \sum_{k=1}^{N} (\bar{Q}_{12})_k (z_k - z_{k-1}) & \sum_{k=1}^{N} (\bar{Q}_{22})_k (z_k - z_{k-1}) & \sum_{k=1}^{N} (\bar{Q}_{26})_k (z_k - z_{k-1}) \\ \sum_{k=1}^{N} (\bar{Q}_{16})_k (z_k - z_{k-1}) & \sum_{k=1}^{N} (\bar{Q}_{26})_k (z_k - z_{k-1}) & \sum_{k=1}^{N} (\bar{Q}_{66})_k (z_k - z_{k-1}) \end{bmatrix}^k \begin{Bmatrix} \varepsilon_y^0 \\ \varepsilon_y^0 \\ \varepsilon_{xy}^0 \end{Bmatrix}^k + \begin{bmatrix} \sum_{k=1}^{N} (\bar{Q}_{11})_k \frac{1}{2}(z_k^2 - z_{k-1}^2) & \sum_{k=1}^{N} (\bar{Q}_{12})_k \frac{1}{2}(z_k^2 - z_{k-1}^2) & \sum_{k=1}^{N} (\bar{Q}_{16})_k \frac{1}{2}(z_k^2 - z_{k-1}^2) \\ \sum_{k=1}^{N} (\bar{Q}_{12})_k \frac{1}{2}(z_k^2 - z_{k-1}^2) & \sum_{k=1}^{N} (\bar{Q}_{22})_k \frac{1}{2}(z_k^2 - z_{k-1}^2) & \sum_{k=1}^{N} (\bar{Q}_{26})_k \frac{1}{2}(z_k^2 - z_{k-1}^2) \\ \sum_{k=1}^{N} (\bar{Q}_{16})_k \frac{1}{2}(z_k^2 - z_{k-1}^2) & \sum_{k=1}^{N} (\bar{Q}_{26})_k \frac{1}{2}(z_k^2 - z_{k-1}^2) & \sum_{k=1}^{N} (\bar{Q}_{66})_k \frac{1}{2}(z_k^2 - z_{k-1}^2) \end{bmatrix}^k \begin{Bmatrix} \kappa_y^0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{Bmatrix}^k$$

A **B**

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} \sum_{k=1}^{N} (\bar{Q}_{11})_k \frac{1}{2}(z_k^2 - z_{k-1}^2) & \sum_{k=1}^{N} (\bar{Q}_{12})_k \frac{1}{2}(z_k^2 - z_{k-1}^2) & \sum_{k=1}^{N} (\bar{Q}_{16})_k \frac{1}{2}(z_k^2 - z_{k-1}^2) \\ \sum_{k=1}^{N} (\bar{Q}_{12})_k \frac{1}{2}(z_k^2 - z_{k-1}^2) & \sum_{k=1}^{N} (\bar{Q}_{22})_k \frac{1}{2}(z_k^2 - z_{k-1}^2) & \sum_{k=1}^{N} (\bar{Q}_{26})_k \frac{1}{2}(z_k^2 - z_{k-1}^2) \\ \sum_{k=1}^{N} (\bar{Q}_{16})_k \frac{1}{2}(z_k^2 - z_{k-1}^2) & \sum_{k=1}^{N} (\bar{Q}_{26})_k \frac{1}{2}(z_k^2 - z_{k-1}^2) & \sum_{k=1}^{N} (\bar{Q}_{66})_k \frac{1}{2}(z_k^2 - z_{k-1}^2) \end{bmatrix}^k \begin{Bmatrix} \varepsilon_y^0 \\ \varepsilon_y^0 \\ \varepsilon_{xy}^0 \end{Bmatrix}^k + \begin{bmatrix} \sum_{k=1}^{N} (\bar{Q}_{11})_k \frac{1}{3}(z_k^3 - z_{k-1}^3) & \sum_{k=1}^{N} (\bar{Q}_{12})_k \frac{1}{3}(z_k^3 - z_{k-1}^3) & \sum_{k=1}^{N} (\bar{Q}_{16})_k \frac{1}{3}(z_k^3 - z_{k-1}^3) \\ \sum_{k=1}^{N} (\bar{Q}_{12})_k \frac{1}{3}(z_k^3 - z_{k-1}^3) & \sum_{k=1}^{N} (\bar{Q}_{22})_k \frac{1}{3}(z_k^3 - z_{k-1}^3) & \sum_{k=1}^{N} (\bar{Q}_{26})_k \frac{1}{3}(z_k^3 - z_{k-1}^3) \\ \sum_{k=1}^{N} (\bar{Q}_{16})_k \frac{1}{3}(z_k^3 - z_{k-1}^3) & \sum_{k=1}^{N} (\bar{Q}_{26})_k \frac{1}{3}(z_k^3 - z_{k-1}^3) & \sum_{k=1}^{N} (\bar{Q}_{66})_k \frac{1}{3}(z_k^3 - z_{k-1}^3) \end{bmatrix}^k \begin{Bmatrix} \kappa_y^0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{Bmatrix}$$

B **D**

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^{o} \\ \varepsilon_y^{o} \\ \gamma_{xy}^{o} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

ABD?

18 terms are governing:

- A = [in-plane stiffness matrix]
- D = [bending stiffness matrix]
- B = [bending-extension coupling matrix]

B=0 if symetrical vs middle plan

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^0{}_x \\ \varepsilon^0{}_y \\ \gamma^0{}_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

Aij, Bij, Dij Coefficients

Aij, Bij, Dij depends on thickness, orientation, stacking and materials properties of each ply.

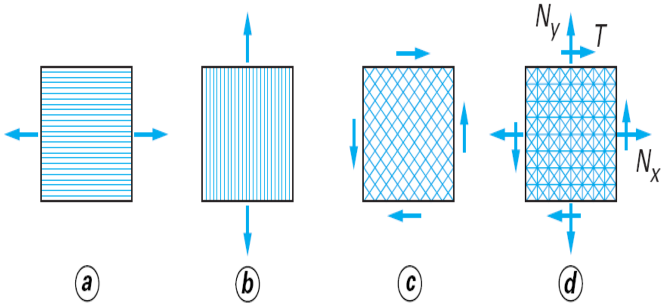
$$A_{ij} = \sum_{k=1}^{N} \left(\overline{Q}_{ij}\right)_k (z_k - z_{k-1})$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{N} \left(\overline{Q}_{ij}\right)_k ({z^2}_k - {z^2}_{k-1})$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N} \left(\overline{Q}_{ij}\right)_k ({z^3}_k - {z^3}_{k-1})$$

where zk are ply coordinate (sup and inf) (i,j=1,2,6)
IS THERE ANY COMPACT FORM OF THIS COMPLEX PROBLEM EXISTING ???????

Optimal Fibers Orientation



Lamination parameters

Lamination parameters are a compact representation of the stacking sequence. Miki, Tsaï and Pagano carried out this representation of the mechanical behaviour of a laminate by decomposing the material-dependent part and the stacking-sequence dependent part, ending up with :

- 5 so-called *material invariants* or *Tsaï-Pagano parameters* $\{U_i\}_{i=1...5}$ that only depend on the material properties (E_{11}, E_{22}, G_{12} and ν_{12} the longitudinal, transerve and shear moduli and the Poisson's ratio)
- 12 so-called *lamination parameters* $\xi^A_{\{1,2,3,4\}}, \xi^B_{\{1,2,3,4\}}, \xi^D_{\{1,2,3,4\}}$ that only depend on the stacking sequence (fiber orientations and number of plies).

Lamination parameters

The knowledge of both U and ξ is enough to derive the constitutive law of the material. More precisely, they allow ud to compute the ABD stiffness tensor where :

- A is the in-plane stiffness tensor that describe the tensile compressive behaviour of the material
- D is the out-of-plane stiffness tensor that describe the flexural behaviour
- B is the coupling stiffness tensor

Lamination parameters

Their definition is :

$$\xi^A_{\{1,2,3,4\}} = \frac{1}{h}\int_{-h/2}^{h/2}\{\cos(2\theta(z)),\cos(4\theta(z)),\sin(2\theta(z)),\sin(4\theta(z))\}dz \quad (1)$$

$$\xi^B_{\{1,2,3,4\}} = \frac{1}{h}\int_{-h/2}^{h/2}\{\cos(2\theta(z)),\cos(4\theta(z)),\sin(2\theta(z)),\sin(4\theta(z))\}zdz \quad (2)$$

$$\xi^D_{\{1,2,3,4\}} = \frac{1}{h}\int_{-h/2}^{h/2}\{\cos(2\theta(z)),\cos(4\theta(z)),\sin(2\theta(z)),\sin(4\theta(z))\}z^2dz \quad (3)$$

where h denotes the thickness of the laminate and θ the orientation of fiber at height $z \in [-h/2, h/2]$. It is a general definition of ξ. For laminate composite it turns down to a simple finite sum of N_{plies} terms

Lamination parameters

Knowing these lamination parameters and the material invariants U. The stiffness tensors can be obtained by

$$\begin{pmatrix} I_{11} \\ I_{22} \\ I_{12} \\ I_{66} \\ I_{16} \\ I_{26} \end{pmatrix} = \begin{pmatrix} 1 & \xi_1^I & \xi_2^I & 0 & 0 \\ 1 & -\xi_1^I & \xi_2^I & 0 & 0 \\ 0 & 0 & -\xi_1^I & 1 & 0 \\ 0 & 0 & -\xi_2^I & 0 & 1 \\ 0 & -\xi_3^I/2 & \xi_4^I & 0 & 0 \\ 0 & \xi_3^I/2 & -\xi_4^I & 0 & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} \tag{4}$$

where I stands for A, B and D.

Lamination parameters based lay-up optimization

The lamination parameters offers another representation of the stacking sequence

- They represent the stacking sequence description through 12 variables, no matter how many plies there are.
- A lot of work has been done on describing the feasible design space for lamination parameters : Miki, Diaconu, Weaver... and the boundaries may be described with a closed form expression. Recently Weaver and al. carried out an implicit relationship to describe to feasible space for any set of arbitrary orientations.
- They can be used to build up surrogate model.

Lamination parameters

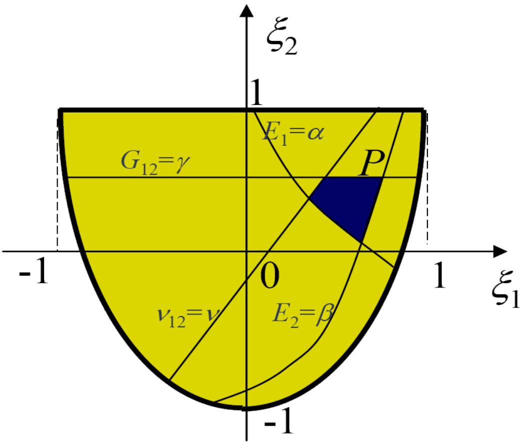


Figure: Exemple of feasible design space into the lamination parameter representation

Lamination parameter based lay-up optimization

This representation has been used with different optimization techniques:

- Evolutionnary techniques : Haftka, Leriche, Todoroki...
- Gradient based techniques : Herencia, Kere
- other metaheuristics (SA, AC...) : Todoroki

Nonetheless, once an optimum has been found in the lamination parameters space, we have to find a corresponding stacking sequence

This is not trivial and in the litterature, this problem is usually solved with evolutionnary techniques. though some author derived original methods : Todoroki used for instance a brounch-and-bound that is based on the fractal structure of the lamination parameters space.

Others representations

Note that this representation is not the only one to be used for lay-up optimization

- We can use directly the reduced stiffness tensors A and D and use continous techniques (gradient descente techniques) (Adams, Herencia,...). Note that we still have the post-identification problem.
- There also exists an equivalent representation used by Vanucci : the polar form