Introduction to Kalman filtering

Course key notes
Corrected exercises
Matlab training session

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Introduction

This document is an introduction to **Kalman optimal filtering** applied to linear systems. It is assumed that the reader is already aware of linear control theory (servo-loop systems), frequency-domain filtering (continuous and discrete-time) and state-space approach to represent linear systems.

Generally, filtering consists in estimating a **useful information** (signal) from a **measurement** (of this information) perturbed by a **noise**. Frequency-domain filtering assumes that a frequency-domain separation exists between the frequency response of the useful signal and the frequency response of the noise. Then, frequency-domain filtering consists in seeking a transfer function fitting a template on its magnitude response (and too much rarely, on its phase response). KALMAN optimal filtering aims to **estimate the state vector** of a linear system (thus, this state is the useful information) and this estimate is optimal w.r.t. the estimation error variances for all state vector components. First of all, some backgrounds on random variables and signals are required then, assumptions, structure and computation of KALMAN filter could be introduced.

In the first chapter, we remind to the reader how a **random signal** can be characterized from a mathematical point of view. The response of a linear system to a random signal will be defined to complete the more well-known response of a linear system to a deterministic signal (impulse, step, ramp, ... responses). In the second chapter, the assumptions, the structure, the main parameters and properties of KALMAN filter will be defined. The reader who wishes to learn practical KALMAN filter tuning methodology can directly start the reading at chapter 2. But the reading of chapter 1, which is more cumbersome from a theoritical point of view, is required if one wishes to learn basic principles in random signal processing, on which is based KALMAN filtering.

There are many applications of KALMAN filtering in aeronautics and aerospace engineering. As KALMAN filter provides an estimate of plant states from an priori information of the plant behaviour (**model**) and from real measurement, KALMAN filter can be used to estimate initial conditions (**ballistics**), to predict vehicle position and trajectory (**navigation**) and also to implement control laws based on a state feedback and a state estimator (**LQG: Linear Quadratic Gaussian con-**

trol). The **signal processing** principles on which is based KALMAN filter will be also very useful to study and perform **test protocols**, experimental data processing and also parametric **identification**, that is the experimental determination of some plant dynamic parameters.

Chapter 1

Random signal in linear systems

1.1 Problem statement

Let us consider a state-space model of a linear and time-invariant plant:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{M}\mathbf{w}(t) & \text{(state equation)} \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{v}(t) & \text{(measurement equation)} \end{cases} \tag{1.1}$$

where :

- $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector of the plant,
- $\mathbf{u}(t) \in \mathbb{R}^m$ is the deterministic input vector (known inputs: control signal, ...),
- $\mathbf{w}(t) \in \mathbb{R}^q$ is the vector of unknown random signals (noises) that perturb the state equation through the input matrix $\mathbf{M}_{n\times q}$ ($\mathbf{w}_\mathbf{x} = \mathbf{M}\mathbf{w}$ denotes also the state noise),
- $\mathbf{y}(t) \in \mathbb{R}^p$ is the measurement vector,
- $\mathbf{v}(t) \in \mathbb{R}^p$ is the vector of random signals (measurements noise) perturbing the measurement (it is assumed there are as many noises as measurements).

Example 1.1 *The "quarter vehicle" model used to study a car active suspension can be depicted by Figure 1.1. The control device (piston, jack) allows a force u to be applied between the wheel (with a mass m and a vertical position z) and the body of the car (with a mass M and a vertical position Z). K and f denote the stiffness and the damping of the passive suspension. k is the tire stiffness between wheel and ground. Finally, w denotes the vertical position of the tire/ground point of contact excited by road unevenness (rolling noise). Body vertical acceleration is measured*

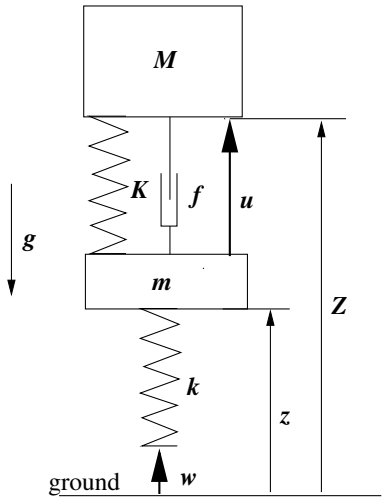

Figure 1.1: Quarter vehicle model.

by the means of an inertial sensor (an accelerometer tacking into account gravity $g = -9.81\, m/s^2$*). The noise on this accelerometer is denoted* v*.*
Let us denote δz *et* δZ *small variations of* z *and* Z *around an equilibrium position* z_0 *et* Z_0 *(equilibrium conditions are:* $w = 0$*,* $u = 0$ *and gravity effects are entirely compensated by stiffness (*k *and* K*) compression). Then applying* NEWTON *principle on both masses* m *and* M*, we get:*

$$\begin{aligned} M\ddot{\delta Z} &= K(\delta z - \delta Z) + f(\dot{\delta z} - \dot{\delta Z}) + u \\ m\ddot{\delta z} &= -K(\delta z - \delta Z) - f(\dot{\delta z} - \dot{\delta Z}) - u - k(\delta z - w)\ . \end{aligned}$$

The measurement equation is : $y = \ddot{\delta Z} + g + v$*.*

Let $\mathbf{x} = [\delta Z,\ \delta z,\ \dot{\delta Z},\ \dot{\delta z}]^T$ *be the state vector, then the following state space representation can be derived:*

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K/M & K/M & -f/M & f/M \\ K/m & -(K+k)/m & f/m & -f/m \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/M & 0 \\ -1/m & 0 \end{bmatrix} \begin{bmatrix} u \\ g \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/k \end{bmatrix} w \\ y &= \begin{bmatrix} -K/M & K/M & -f/M & f/M \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1/M & 1 \end{bmatrix} \begin{bmatrix} u \\ g \end{bmatrix} + v\ . \end{aligned} \tag{1.2}$$

This model is in the form of (1.1) with $n = 4$*,* $m = 2$*,* $q = 1$ *and* $p = 1$*.*

□

We recognize in matrices ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$) of model (1.1) the well known state space representation of the transfer between deterministic input $\mathbf{u}$ and measurement

$\mathbf{y}$: [1] :

$$\mathbf{F}(\mathrm{s}) = \mathbf{D} + \mathbf{C}(\mathrm{s}1 - \mathbf{A})^{-1}\mathbf{B} \; .$$

The response of this model to a deterministic input $\mathbf{u}(t)$ over a time range $t \in [t_0, \; t]$ and to initial conditions $\mathbf{x}(t_0)$ reads :

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} \mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\, d\tau \tag{1.3}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{1.4}$$

Proof: see appendix A where an illustrative exercise is also proposed.

What about the response of the model (1.1) to a random signal $\mathbf{w}(t)$?

To answer this question, it is required:

- firstn to remind how a random signal can be characterized from a mathematical (or stochastic) point of view,
- to add some assumptions on the stochastic properties of noises $w(t)$ and $v(t)$ (gaussian white noise) to make the determination of the response of model 1.1 easier,
- to compute stochastic characteristics of the response of $x(t)$ and $y(t)$.

The various definitions and background given below are extracted from reference [5] (chapter II and appendix A.I).

1.2 Background and definitions

1.2.1 Characterization of random variables

Let $\mathcal{X}$ be a random variable defined on the real space $\mathbb{R}$. The **probability distribution function** $F(x)$ associates, to each real value x, the probability of the occurrence $\mathcal{X} < x$. That is:

$$F : \mathbb{R} \to [0 \quad 1] \quad / \quad F(x) = \Pr[\mathcal{X} < x] \; .$$

Properties:

- $\forall x_1 < x_2, \quad \Pr[x_1 \leq \mathcal{X} < x_2] = F(x_2) - F(x_1)$,
- $\lim_{x \to +\infty} F(x) = 1$; $\lim_{x \to -\infty} F(x) = 0$.

[1] s stands for LAPLACE variable.

- $F(x)$ is a monotonous, increasing function, and can be continuous or discontinuous depending on $\mathcal{X}$ has continuous or discrete values, respectively.

If $F(x)$ is differentiable, then its derivative w.r.t. x is called **probability density function** and is denoted $p(x)$:

$$p(x) = \frac{dF(x)}{dx} \quad \text{soit :} \quad p(x)dx = \Pr[x \leq \mathcal{X} < x + dx] \ .$$

To characterize a random variable $\mathcal{X}$, one can also used the **moments** of this variable. The first **moment** is called **mean value** or **expected value**. The second **central moment** is called **variance** and is denoted $var_x = \sigma_x^2$ where σ_x is the **standard deviation**. That is:

- **expected value**:

$$\mathrm{E}[\mathcal{X}] = \int_{-\infty}^{+\infty} x\, p(x)\, dx = \int_{-\infty}^{+\infty} x\, dF(x) \ , \tag{1.5}$$

- k**-th moment**:

$$\mathrm{E}[\mathcal{X}^k] = \int_{-\infty}^{+\infty} x^k\, p(x)\, dx \ , \tag{1.6}$$

- k**-th central moment**: [2]

$$\mathrm{E}[(\mathcal{X} - \mathrm{E}[\mathcal{X}])^k] = \int_{-\infty}^{+\infty} (x - \mathrm{E}[\mathcal{X}])^k\, p(x)\, dx \ .$$

Full description of a random variable requires the characterization of all its moments. But from a practical point of view, 3-th and higher moments are not used because they cannot be computed or derived easily. The (mathematical) interest of normal (**gaussian**) random variable lies is the fact they are entirely defined by the first and second moments. Indeed, let $\mathcal{X}$ a **gaussian** random variable with a mean value m and a standard deviation σ, then:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(x-m)^2}{2\sigma^2}}, \quad \mathrm{E}[x] = m, \quad \mathrm{E}[(x-m)^2] = \sigma^2 \ .$$

Example 1.2 *[Uniform distribution] The probability density function of the random variable $\mathcal{X}$ is constant between two values a and b with $b > a$.*

$$\mathrm{E}[\mathcal{X}] = \int_a^b \frac{x}{b-a}\, dx = \frac{1}{b-a} \left[\frac{1}{2}x^2\right]_a^b = \frac{1}{2}\frac{b^2-a^2}{b-a} = \frac{a+b}{2}$$

[2] Recall: $var_x = \mathrm{E}[\mathcal{X}^2] - \mathrm{E}[\mathcal{X}]^2$.

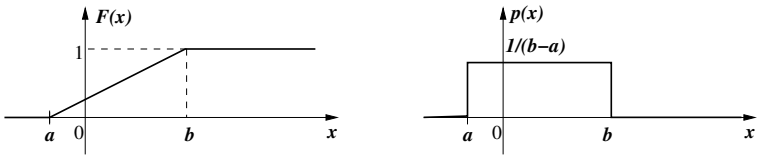

Figure 1.2: Probability distribution and density functions for a uniform distribution.

$$var_x = \mathrm{E}[(\mathcal{X} - \mathrm{E}[\mathcal{X}])^2] = \frac{1}{b-a}\int_a^b \left(x - \frac{a+b}{2}\right)^2 dx = \frac{1}{b-a}\left[\frac{1}{3}\left(x - \frac{a+b}{2}\right)^3\right]_a^b$$

$$\Rightarrow var_x = \frac{1}{3}\frac{1}{b-a}\left(\frac{b-a}{2}\right)^3 = \frac{(b-a)^2}{12} .$$

□

Discrete random variable: If a random variable is defined on a set of N discrete values $x_i,\ i = 1, \cdots, N$ then the density function is no more used and one can directly define the "probability that $\mathcal{X} = x_i$" noted $\Pr[\mathcal{X} = x_i]$. The definition of moments involves a discrete sum :

$$\mathrm{E}[\mathcal{X}^k] = \sum_{i=1}^{N} x_i^k \Pr[\mathcal{X} = x_i] .$$

Example 1.3 *The random variable $\mathcal{X}$ corresponding to a dice tossing:*

$$\Pr[\mathcal{X} = i] = \frac{1}{6},\ \forall\, i = 1, \cdots, 6\ ; \quad \mathrm{E}[\mathcal{X}] = \frac{21}{6}\ ; \quad var_x = \frac{35}{12} .$$

□

1.2.2 Caracterization of multivariate random variables

Let $\mathcal{X} = [\mathcal{X}_1, \cdots, \mathcal{X}_q]^T$ a vectorial random variable with q components taking on the values in $\mathbb{R}^q$.

Probability distribution function

(i.e. the joint distribution of the q random variables $\mathcal{X}_i,\ i = 1, \cdots q$)

$$F(x_1, \cdots, x_q) = \Pr[\mathcal{X}_1 < x_1 \text{ and } \mathcal{X}_2 < x_2 \text{ and } \cdots \text{ and } \mathcal{X}_q < x_q .$$

Density function

$$p(x_1, \cdots, x_q) = \frac{\partial^q F(x_1, \cdots, x_q)}{\partial x_1 \cdots \partial x_q} \ .$$

Moments

Define $\mathbf{x} = [x_1, \cdots, x_q]^T$. Only the vector of first moments (that is the **mean vector**) and the matrix of second central moments (that is the **covariance matrix**) are considered:

- mean: $\mathrm{E}[\mathcal{X}] = [\mathrm{E}[\mathcal{X}_1], \cdots, \mathrm{E}[\mathcal{X}_q]]^T$.
- covariance: $\mathrm{Cov}_{\mathbf{x}} = \mathrm{E}[(\mathcal{X} - \mathrm{E}[\mathcal{X}])(\mathcal{X} - \mathrm{E}[\mathcal{X}])^T]$. The component $\mathrm{Cov}_{\mathbf{x}}(i,j)$ at row i and column j of this covariance matrix verifies:

$$\mathrm{Cov}_{\mathbf{x}}(i,j) = \int_{\mathbb{R}^2} (x_i - \mathrm{E}[\mathcal{X}_i])(x_j - \mathrm{E}[\mathcal{X}_j]) \, dF(x_i, x_j) \ .$$

  The covariance matrix is definite, positive and symmetric.

Gaussian random vector with mean m and covariance Δ

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{q/2}\sqrt{\det \mathbf{\Delta}}} \mathrm{e}^{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \mathbf{\Delta}^{-1}(\mathbf{x}-\mathbf{m})} \ .$$

The gaussian random vector $\mathcal{X}$ with mean $\mathbf{m}$ and covariance $\mathbf{\Delta}$ can be build from the **normal gaussian vector** $\mathcal{N}$ (that is the vector with a null mean and an unitary covariance) in the following way:

$$\mathcal{X} = \mathbf{m} + \mathbf{G}\mathcal{N}$$

where $\mathbf{G}$ is such that: $\mathbf{G}\mathbf{G}^T = \mathbf{\Delta}$.

Independence

Two random variables $\mathcal{X}_1$ and $\mathcal{X}_2$ are independent if and only if:

$$F(x_1, x_2) = F(x_1)F(x_2) \ .$$

An independence necessary condition is:

$$\mathrm{E}[\mathcal{X}_1\mathcal{X}_2] = \mathrm{E}[\mathcal{X}_1]\mathrm{E}[\mathcal{X}_2] \ . \tag{1.7}$$

Exercise 1.1 *Lest us consider 2 independent random variables $\mathcal{X}_1$ and $\mathcal{X}_2$ with a uniform density between -1 and 1 for each of these 2 variables. Define a new random variable as $\mathcal{Y} = \frac{\mathcal{X}_1+\mathcal{X}_2}{2}$.*
Compute $\mathrm{E}[\mathcal{Y}]$, var_y and $cov_{x_1,y}$ the covariance matrix of the vector: $(\mathcal{X}_1,\ \mathcal{Y})^T$.

Correction: *From exercise 1.2, one can derive:*

$$\mathrm{E}[\mathcal{X}_1] = \mathrm{E}[\mathcal{X}_2] = 0; \quad var_{x_1} = var_{x_2} = \frac{1}{3}\ .$$

By definition: $\mathcal{Y} = (1/2 \quad 1/2)\begin{pmatrix}\mathcal{X}_1\\ \mathcal{X}_2\end{pmatrix}$. *Then :*

$$\mathrm{E}[\mathcal{Y}] = (1/2 \quad 1/2)\, E\left[\begin{pmatrix}\mathcal{X}_1\\ \mathcal{X}_2\end{pmatrix}\right] = 0\ ,$$

$$var_y = (1/2 \quad 1/2) E\left[\begin{pmatrix}\mathcal{X}_1\\ \mathcal{X}_2\end{pmatrix}(\mathcal{X}_1 \quad \mathcal{X}_2)\right]\begin{pmatrix}1/2\\ 1/2\end{pmatrix} = (1/2 \quad 1/2) cov_{x_1,x_2}\begin{pmatrix}1/2\\ 1/2\end{pmatrix}$$

$\mathcal{X}_1$ *and* $\mathcal{X}_2$ *are independent, thus:* $cov_{x_1,x_2} = \begin{pmatrix}1/3 & 0\\ 0 & 1/3\end{pmatrix}$

$$\Rightarrow \quad var_y = \frac{1}{6}\ .$$

In the same manner:

$$\begin{aligned} cov_{x_1,y} &= \mathbf{G} cov_{x_1,x_2}\mathbf{G}^T, \quad with:\ \mathbf{G} = \begin{pmatrix}1 & 0\\ 1/2 & 1/2\end{pmatrix}\\ &= \begin{pmatrix}1/3 & 1/6\\ 1/6 & 1/6\end{pmatrix}\end{aligned}\ .$$

Remark : At the cost of more tedious calculus, one can also give a full caracterization of the random variable $\mathcal{Y}$ by its distribution function $F(y)$ or density function $p(y)$ and then compute first and second moments:

$$F(y) = \Pr\left[\frac{\mathcal{X}_1+\mathcal{X}_2}{2} < y\right] = \Pr[\mathcal{X}_1 < 2y - \mathcal{X}_2] = \int_{\mathcal{D}_{x_2}} \Pr[\mathcal{X}_1 < 2y - x_2]p(x_2)dx_2\ .$$

For a given value y, we can write (see Figure 1.2):

$$\begin{aligned}
\Pr[\mathcal{X}_1 < 2y - x_2] &= 0, \quad \forall\ x_2\ /\ 2y - x_2 < -1\ \Rightarrow\ \forall\ x_2 > 2y + 1,\\
\Pr[\mathcal{X}_1 < 2y - x_2] &= \frac{2y - x_2 + 1}{2}, \quad \forall\ x_2\ /\ -1 \le 2y - x_2 < 1\ \Rightarrow\ \forall\ x_2\ /\ 2y - 1 < x_2 \le 2y + 1,\\
\Pr[\mathcal{X}_1 < 2y - x_2] &= 1, \quad \forall\ x_2\ /\ 2y - x_2 \ge 1\ \Rightarrow\ \forall\ x_2 \le 2y - 1.
\end{aligned}$$

And $p(x_2) = 1/2$ *if* $-1 \leq x_2 < 1$, $p(x_2) = 0$ *else. Then* [3]*:*

- *if* $y < 0 \quad F(y) = \int_{2y+1}^{1} 0\, \frac{1}{2} dx_2 + \int_{-1}^{2y+1} \frac{2y-x_2+1}{4} dx_2 + 0 = \frac{(y+1)^2}{2}$,
- *if* $y \geq 0 \quad F(y) = 0 + \int_{2y-1}^{1} \frac{2y-x_2+1}{4} dx_2 + \int_{-1}^{2y-1} \frac{1}{2} dx_2 = \frac{-y^2+2y+1}{2}$.

That yields to:

$$
\begin{array}{l}
p(y) = y + 1 \quad \forall y \in [-1,\ 0], \\
p(y) = -y + 1 \quad \forall y \in [0,\ 1], \\
\mathrm{E}[\mathcal{Y}] = \int_{-1}^{1} y\, p(y) dy = \int_{-1}^{0} y(y+1) dy + \int_{0}^{1} y(-y+1) dy = 0, \\
var_y = \int_{-1}^{1} y^2 p(y) dy = \int_{-1}^{0} y^2 (y+1) dy + \int_{0}^{1} y^2 (-y+1) dy = \frac{1}{6} .
\end{array}
$$

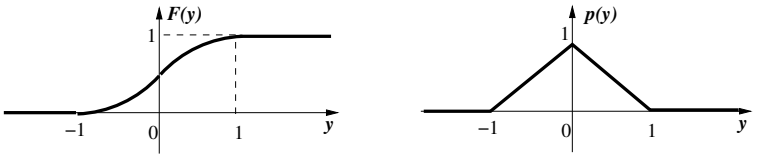

Figure 1.3: Distribution and density functions of $\mathcal{Y}$.

□

In the following, random variable (resp. signal) stands for both single ($q = 1$) or multivariate (q components) variable (resp. signal).

1.2.3 Random signal (or process)

Given a random variable $\mathcal{X}$, the **random signal** $\mathbf{x}(t)$ is a function of time t such that for each given t, $\mathbf{x}(t)$ corresponds to a value (a sample) of $\mathcal{X}$.

[3] Using conditional probabilities, one can also write:

$$
\begin{aligned}
F(y) \quad = \quad & \int_{\mathcal{D}_{x_2}} 0\, p(x_2 | x_2 > 2y+1) dx_2 + \int_{\mathcal{D}_{x_2}} \frac{2y - x_2 + 1}{2} p(x_2 | 2y-1 < x_2 \leq 2y+1) dx_2 \\
+ \quad & \int_{\mathcal{D}_{x_2}} 1\, p(x_2 | x_2 \leq 2y-1) dx_2 \ .
\end{aligned}
$$

1.2.4 Moments for random signal

The second **moment** of a random signal is called the **auto-correlation function**.

Let $\mathbf{w}(t)$ be a random signal, then:

$$\text{first moment:} \quad \mathbf{m}(t) = \mathrm{E}[\mathbf{w}(t)] \tag{1.8}$$

$$\text{second moment:} \quad \boldsymbol{\phi}_{\mathbf{ww}}(t,\tau) = \mathrm{E}[\mathbf{w}(t)\mathbf{w}(t+\tau)^T]\,. \tag{1.9}$$

Remark 1.1 *if $\mathbf{w}(t)$ is a vectorial (multivariate) signal with q components then $\boldsymbol{\phi}_{\mathbf{ww}}(t,\tau)$ is a $q \times q$ definite positive matrix for each value of t and τ. The diagonal terms are the scalar auto-correlation functions of each components and the cross terms are the **inter-correlation** functions between components.*

A **centered gaussian random signal**, that is a signal generated at each instant from a sample of a gaussian random variable with a null mean value, is therefore entirely defined by its auto-correlation function.

1.2.5 Stationarity

A random signal is defined to be **wide sense stationary** [4] if its mean is constant ($\mathbf{m}(t) = \mathbf{m}$) and if its auto-correlation function depends only on τ ($\boldsymbol{\phi}_{\mathbf{ww}}(t,\tau) = \boldsymbol{\phi}_{\mathbf{ww}}(\tau)$).

The quadratic mean (or variance [5] if the signal is centered) of a stationary random signal is the auto-correlation function value at the origin:

$$\mathbf{var}_{\mathbf{w}} = \boldsymbol{\phi}_{\mathbf{ww}}(\tau)|_{\tau=0}$$

Exercise 1.2 *A random signal $b(t)$ is generated in the following way: from the initial time $t_0 = 0$, the signal $b(t)$ is hold every dt seconds on the value (sample) of a gaussian centered random variable $\mathcal{X}$ with a standard deviation σ; all samples x_i of the variable $\mathcal{X}$ are independent to each others. So: $b(t) = x_i\ \forall t \in [idt,\ (i+1)dt[$ (see Figure 1.4).*

- *Compute the mean $m(t)$ and the auto-correlation function $\phi_{bb}(t,\tau)$ of the signal $b(t)$. Is $b(t)$ wide sense stationary ?.*
- *Answer again previous questions assuming now that the initial instant is a random variable with a uniform distribution between 0 and dt.*

Solution keys:

[4] One can also defined the strict sense stationary (see reference[3]).

[5] or covariance matrix in the case of a vectorial random signal.

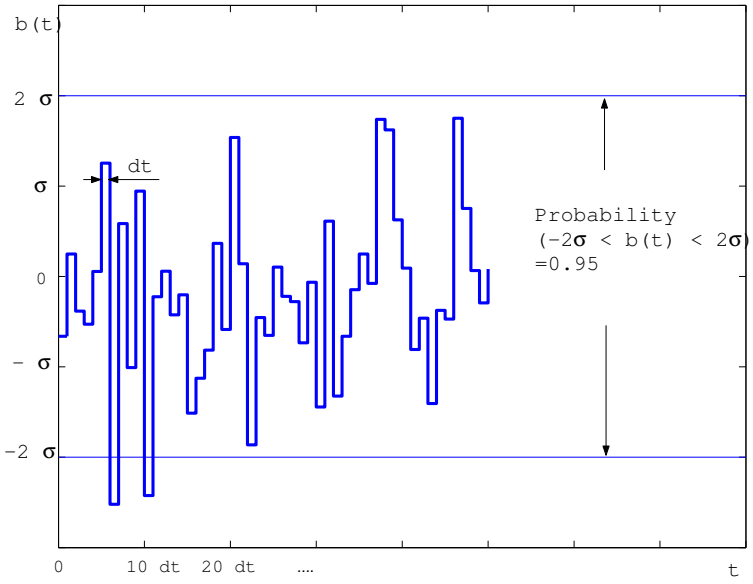

Figure 1.4: Time domain response of a realization of random signal $b(t)$.

- $m(t) = \mathrm{E}[b(t)] = \mathrm{E}[\mathcal{X}] = 0, \quad \forall t,$
- $\phi_{bb}(t,\tau) = \mathrm{E}[b(t)b(t+\tau)] = \sigma^2$ *if* t *and* $t+\tau$ *are in the same time interval* dt*;* 0 *else (because the samplings are independent and centered).*
- *the auto-correlation function depends on* t *and* τ*; so* $b(t)$ *is not a stationary signal. For instance, for* $t = i\,dt + \varepsilon$ $(\forall i, \forall \epsilon \in [0,\ dt[)$ *, the response of* $\phi_{bb}(t,\tau)$ *is plotted in Figure 1.5.*
- *if initial time* t_0 *is now a random variable with a uniform distribution taking value between 0 and* dt *(let* $b'(t)$ *be this new random signal) then one can consider in the previous calculus that* ε *has a uniform density between 0 and* dt*:*

$$\phi_{b'b'}(t,\tau) = \frac{1}{dt}\int_0^{dt} \phi_{bb}(i\,dt + \varepsilon, \tau)d\varepsilon$$

  - *for* $0 \leq \tau < dt$*,* $\phi_{bb}(i\,dt + \varepsilon, \tau) = \sigma^2$ *iff* $0 < \varepsilon < dt - \tau$*,* 0 *otherwise. That is:*

$$\phi_{b'b'}(t,\tau) = \frac{\sigma^2}{dt}\int_0^{dt-\tau} d\varepsilon = \frac{\sigma^2}{dt}(dt-\tau) \quad \forall \tau \ / \ 0 \leq \tau < dt\ .$$

  - *for* $-dt < \tau < 0$*,* $\phi_{bb}(i\,dt + \varepsilon, \tau) = \sigma^2$ *iff* $-\tau < \varepsilon < dt$*,* 0 *otherwise. That is:*

$$\phi_{b'b'}(t,\tau) = \frac{\sigma^2}{dt}\int_{-\tau}^{dt} d\varepsilon = \frac{\sigma^2}{dt}(dt+\tau) \quad \forall \tau \ / \ -dt < \tau < 0\ .$$

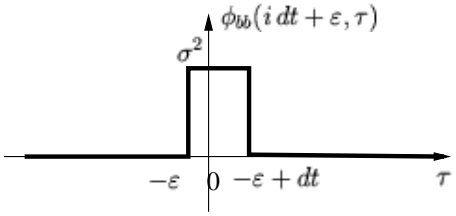

Figure 1.5: Autocorrelation function of signal $b(t)$.

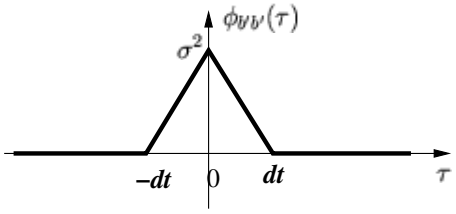

Figure 1.6: Autocorrelation function of signal $b'(t)$.

So $\phi_{b'b'}(t,\tau)=\sigma^2(1-\frac{|\tau|}{dt})\ \ \forall\tau\ \in[-dt\quad dt]$, 0 *else (see Figure 1.6) and depends only on* τ, $b'(t)$ *is now wide sense stationary.*

□

1.2.6 Spectrum in the s-plane

Wide sense stationary random signals can also be characterized by their frequency-domain representation called **Power Spectral Density (PSD)** or also their **spectrum in the s-plane** (the PSD is given by the spectrum where s is replaced by $j\omega$). The spectrum in the s-plane of a wide sense stationary random signal is the bilateral LAPLACE transform of the auto-correlation function [6].

$$\mathbf{\Phi_{ww}}(\mathrm{s})=\mathcal{L}_{II}\left[\boldsymbol{\phi}_{\mathbf{ww}}(\tau)\right]=\int_{-\infty}^{\infty}\boldsymbol{\phi}_{\mathbf{ww}}(\tau)\mathrm{e}^{-\tau\mathrm{s}}d\tau\ . \tag{1.10}$$

Power Spectral Density (PSD): $\mathbf{\Phi_{ww}}(\omega)=\mathbf{\Phi_{ww}}(\mathrm{s})|_{\mathrm{s}=j\omega}$.

[6] The inverse bilateral LAPLACE transform inverse is defined by:

$$\phi_{\mathbf{ww}}(\tau)=\mathcal{L}_{II}^{-1}\mathbf{\Phi_{ww}}(\mathrm{s})=\frac{1}{2\pi\,j}\int_{-j\infty}^{+j\infty}\mathbf{\Phi_{ww}}(\mathrm{s})\mathrm{e}^{\mathrm{s}\tau}ds$$

Remark 1.2 :

$$\begin{aligned}\boldsymbol{\Phi}_{\mathbf{ww}}(\mathrm{s}) &= \int_0^\infty \boldsymbol{\phi}_{\mathbf{ww}}(\tau)\mathrm{e}^{-\tau \mathrm{s}}d\tau + \int_0^\infty \boldsymbol{\phi}_{\mathbf{ww}}(-u)\mathrm{e}^{u\mathrm{s}}du \\ &= \boldsymbol{\Phi}^+_{\mathbf{ww}}(\mathrm{s}) + \boldsymbol{\Phi}^-_{\mathbf{ww}}(-\mathrm{s})\end{aligned}$$

with :

$$\boldsymbol{\Phi}^+_{\mathbf{ww}}(\mathrm{s}) = \mathcal{L}[\boldsymbol{\phi}^+_{\mathbf{ww}}(\tau)] \quad \textit{and} \quad \boldsymbol{\phi}^+_{\mathbf{ww}}(\tau) = \boldsymbol{\phi}_{\mathbf{ww}}(\tau) \textit{ if } \tau \geq 0, \textit{ else } \boldsymbol{\phi}^+_{\mathbf{ww}}(\tau) = 0 \ ,$$

$$\boldsymbol{\Phi}^-_{\mathbf{ww}}(\mathrm{s}) = \mathcal{L}[\boldsymbol{\phi}^-_{\mathbf{ww}}(\tau)] \quad \textit{and} \quad \boldsymbol{\phi}^-_{\mathbf{ww}}(\tau) = \boldsymbol{\phi}_{\mathbf{ww}}(-\tau) \textit{ if } \tau \geq 0, \textit{ else } \boldsymbol{\phi}^-_{\mathbf{ww}}(\tau) = 0 \ .$$

If the function $\boldsymbol{\phi}_{\mathbf{ww}}(\tau)$ is pair then $\Phi^+_{\mathbf{ww}} = \Phi^-_{\mathbf{ww}}$ and the spectrum in the s*-plane $\boldsymbol{\Phi}_{\mathbf{ww}}(\mathrm{s})$ is a function of s^2 (that is, the PSD is real).*

The initial value theorem of the mono-lateral LAPLACE *transform allows the variance to be computed directly from the spectrum in the* s*-plane:*

$$\mathbf{var}_{\mathbf{w}} = \boldsymbol{\phi}_{\mathbf{ww}}(\tau)|_{\tau=0} = \lim_{\mathrm{s}\to\infty} \mathrm{s}\boldsymbol{\Phi}^+_{\mathbf{ww}}(\mathrm{s}) \ .$$

□

Remark 1.3 : *From the PSD and the inverse* LAPLACE *transform, one can write:*

$$\boldsymbol{\phi}_{\mathbf{ww}}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \boldsymbol{\Phi}_{\mathbf{ww}}(\omega)\mathrm{e}^{j\omega\tau}d\omega \quad \textit{and} \quad \mathbf{var}_{\mathbf{w}} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \boldsymbol{\Phi}_{\mathbf{ww}}(\omega)d\omega \ .$$

The variance of the noise $\mathbf{w}$ *is (with a factor $1/2\pi$) the integral of the PSD in the frequency domain.*

□

Example 1.4 *Let us consider again the signal $b'(t)$ of section 1.2 whose autocorrelation function is: $\phi_{b'b'}(t,\tau) = \sigma^2(1 - \frac{|\tau|}{dt})$ (pair function). The spectrum in the* s*-plane of this signal is:*

$$\Phi_{b'b'}(\mathrm{s}) = \int_{-\infty}^{\infty} \sigma^2(1 - \frac{|\tau|}{dt})\mathrm{e}^{-\tau \mathrm{s}}d\tau = \Phi^+_{b'b'}(\mathrm{s}) + \Phi^+_{b'b'}(-\mathrm{s}) \quad \textit{with:}$$

$$\begin{aligned}\Phi^+_{b'b'}(\mathrm{s}) &= \int_0^{dt} \sigma^2(1 - \frac{\tau}{dt})\mathrm{e}^{-\tau \mathrm{s}}d\tau \\ &= \frac{\sigma^2}{dt}\left\{\left[\frac{dt - \tau}{-\mathrm{s}}\mathrm{e}^{-\tau \mathrm{s}}\right]_0^{dt} - \frac{1}{s}\int_0^{dt} \mathrm{e}^{-\tau \mathrm{s}}d\tau\right\} \quad \textit{(integration by parts)} \\ &= \frac{\sigma^2}{dt}\left\{\frac{dt}{\mathrm{s}} + \left[\frac{\mathrm{e}^{-\tau\,\mathrm{s}}}{\mathrm{s}^2}\right]_0^{dt}\right\} = \frac{\sigma^2}{dt\,\mathrm{s}^2}\left(\mathrm{s}\,dt + \mathrm{e}^{-\mathrm{s}\,dt} - 1\right) \ .\end{aligned}$$

$$\boxed{\Phi_{b'b'}(\mathrm{s}) = \frac{\sigma^2}{dt\,\mathrm{s}^2}\left(\mathrm{e}^{-\mathrm{s}\,dt} + \mathrm{e}^{s\,dt} - 2\right)}\,.$$

Thus the PSD is:

$$\Phi_{b'b'}(\omega) = -\frac{\sigma^2}{dt\,\omega^2}\left(\mathrm{e}^{-j\omega\,dt} + \mathrm{e}^{j\omega\,dt} - 2\right) = \frac{2\sigma^2}{dt\,\omega^2}(1 - cos(\omega\,dt)) = \frac{4\sigma^2}{dt\,\omega^2}sin^2(\frac{\omega\,dt}{2})$$

$$\boxed{\Phi_{b'b'}(\omega) = \sigma^2\,dt\frac{sin^2(\frac{\omega\,dt}{2})}{(\frac{\omega\,dt}{2})^2}}\,.$$

$\Phi_{b'b'}(\omega)$ *is plotted in Figure 1.7* [7]*. From a practical point of view, the logarithmic scale representation given in Figure 1.8 is preferred and highlights that the PSD can be considered as constant for frequencies very lower than the sampling frequency* $2\pi/dt$*.* ***This signal can be used as a white noise signal with a constant PSD*** R ***in a given frequency range if*** $2\,\pi/dt$ ***is chosen very large w.r.t. this frequency range and if*** $\boxed{\sigma^2 = R/dt}$ *(see next section).*

□

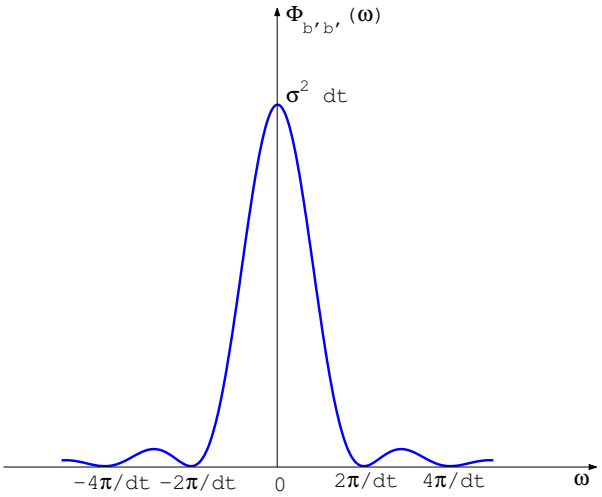

Figure 1.7: PSD for signal $b'(t)$.

This frequency-domain representation of random signal is particularly useful to study and characterized the transmission of random signals in **time-invariant linear systems** (see section 1.3.2).

[7] It is recalled that $\lim_{x\to 0}(\frac{sin\,x}{x}) = 1$

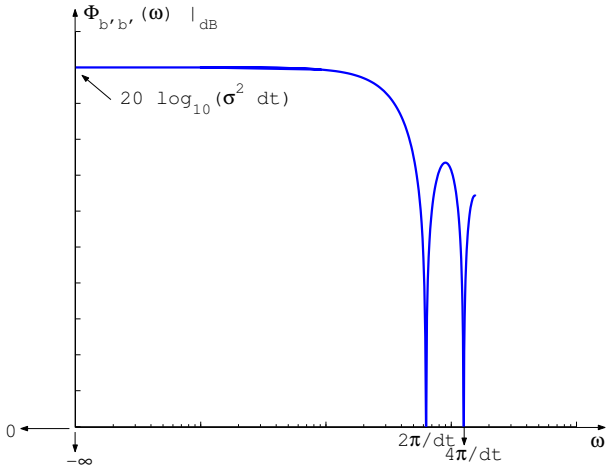

Figure 1.8: PSD for signal $b'(t)$ (logarithmic scales).

1.2.7 White noise

Lastly, a **white noise** is a random signal with an infinite variance and whose auto-correlation function is proportional to a DIRAC distribution $\delta(\tau)$ (that is a constant PSD for any frequency ω). This property expresses that two values of the signal, even sampled at two very closed instants, are not at all correlated.

Central gaussian white noise $w(t)$ and $v(t)$ we are going to used in KALMAN filtering are then entirely defined by their respective **PSD** $\mathbf{W}(t)$ and $\mathbf{V}(t)$:

$$\mathrm{E}[\mathbf{w}(t)\mathbf{w}(t+\tau)^T] = \mathbf{W}(t)\delta(\tau), \quad \mathrm{E}[\mathbf{v}(t)\mathbf{v}(t+\tau)^T] = \mathbf{V}(t)\delta(\tau) \tag{1.11}$$

Matrices $\mathbf{W}(t)$ and $\mathbf{V}(t)$ become constant in the case of stationary gaussian white noises. The normalized gaussian white noise is such that $\mathbf{W}(t) = \mathbf{1}_q$ (q components in the noise).[8].

Remark 1.4 *From a practical point of view, it is not possible to simulate, on a numerical computer, a perfect continuous-time white noise characterized by a finite PSD* $\mathbf{R}$ *(but with an infinite variance). The approximation proposed in example 1.2 (see Figure 1.8), which consists in holding, over a sample period dt (which must be chosen very low w.r.t. the settling time of the system), a gaussian random variable with a variance* $\mathbf{var} = \mathbf{R}/dt$*, will be used. This approximation corresponds to the "Band-limited white-noise" proposed in Matlab/Simulink.*

[8] $\mathbf{1}_q$ is the identity matrix of size q.

1.3 Transmission of random signals in linear systems

1.3.1 Time-domain approach

The assumptions:

- the model is linear (equation 1.1),
- and $\mathbf{w}$ is a centered gaussian white noise

allows to assert that the state $\mathbf{x}$ and the output $\mathbf{y}$ are also gaussian multivariate signals and are therefore entirely characterized by first and second moments. The following theorem will allow these characteristics to be computed. It is assumed that the deterministic input $\mathbf{u}$ is null ($\mathbf{u}(t) = \mathbf{0}$).

Theorem 1.1 *let us consider the linear system:*

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{M}\mathbf{w}(t) \; . \tag{1.12}$$

$\mathbf{w}(t)$ *is a centered stationary Gaussian white noise with a PSD* $\mathbf{W}$. $\mathbf{m}(t_0)$ *et* $\mathbf{P}(t_0)$ *are the mean value and the covariance matrix of the initial state* $\mathbf{x}(t_0)$ *(also a gaussian random variable). Then* $x(t)$ *is a gaussian random signal with:*

- *mean vector:*
$$\mathbf{m}(t) = \mathrm{E}[\mathbf{x}(t)] = \mathrm{e}^{\mathbf{A}(t-t_0)}\mathbf{m}(t_0)$$
- *covariance matrix* $\mathbf{P}(t) = \mathrm{E}[(\mathbf{x}(t) - \mathbf{m}(t))(\mathbf{x}(t) - \mathbf{m}(t))^T]$ *solution of the differential* LYAPUNOV *equation:*
$$\dot{\mathbf{P}}(t) = \mathbf{A}\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}^T + \mathbf{M}\mathbf{W}\mathbf{M}^T \; . \tag{1.13}$$
If the system is stable (all the eigenvalues of $\mathbf{A}$ *have a negative real part), then a steady state (stationary) response can be reached :* $\dot{\mathbf{P}} = \mathbf{0}_n$ *and* $\mathbf{P}(t) = \mathbf{P}$ *solves the continuous-time* LYAPUNOV *equation :*
$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T + \mathbf{M}\mathbf{W}\mathbf{M}^T = 0 \; . \tag{1.14}$$

Proof: (see annex B.2).

Remark 1.5 *If the output equation is considered without noise i.e.* $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$ *then the covariance matrix* $\mathbf{P}_\mathbf{y}(t)$ *of* $\mathbf{y}(t)$ *is:*

$$\mathbf{P}_\mathbf{y}(t) = \mathbf{C}\mathbf{P}(t)\mathbf{C}^T$$

(if the white noise on the measurement is considered i.e. $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + v(t)$*, then the covariance of* $\mathbf{y}$ *is infinite).*

1.3.2 Frequency-domain (spectral) approach

Only the steady state is considered here.

Theorem 1.2 (Transmission of a white noise in the linear system) *Let us consider a* ***stable*** *linear continuous-time system defined by the transfer matrix* $\mathbf{G}(\mathrm{s})_{p\times q}$ *between the input* $\mathbf{w}$ *and the output* $\mathbf{y}$*. The steady state response* $\mathbf{y}$ *to a stationary random signal* $\mathbf{w}$*, characterized by a spectrum* $\mathbf{\Phi_{ww}}(\mathrm{s})_{q\times q}$ *in the* s*-plane, is a stationary random signal characterized by a spectrum* $\Phi_{\mathbf{yy}}(\mathrm{s})$ *in the* s*-plane such that:*

$$\mathbf{\Phi_{yy}}(\mathrm{s})_{p\times p} = \mathbf{G}(-\mathrm{s})\mathbf{\Phi_{ww}}(\mathrm{s})\mathbf{G}^T(\mathrm{s}) \ .$$

Proof: (voir annexe B.3).

In the case of a SISO (Single Input Single Output) transfer G, this result can be depicted by Figure 1.9.

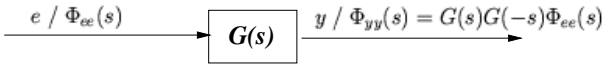

Figure 1.9: SISO case.

Reciprocally : Let us consider a random signal $\mathbf{w}(t)$ with a given real [9] spectrum in the s-plane $\mathbf{\Phi_{ww}}(\mathrm{s})$, then the decomposition (such a decomposition in not unique as it is shown in example 1.3):

$$\mathbf{\Phi_{ww}}(\mathrm{s}) = \mathbf{G}(-\mathrm{s})\mathbf{G}^T(\mathrm{s})$$

where $\mathbf{G}(\mathrm{s})$ is the transfer gathering all the stable poles of $\mathbf{\Phi_{ww}}(\mathrm{s})$, allows to provide a **Marlov model** of the signal $\mathbf{w}(t)$, i.e. a state space realization of the "generator" filter $\mathbf{G}(\mathrm{s})$. This state space realization is denoted ($\mathbf{G}(\mathrm{s})$ is assumed to be strictly proper):

$$\begin{cases} \dot{\mathbf{x}}_\mathbf{G}(t) = \mathbf{A_G}\mathbf{x_G}(t) + \mathbf{B_G}\mathbf{b}(t) \\ \mathbf{w}(t) = \mathbf{C_G}\mathbf{x_G}(t) \end{cases}$$

where $\mathbf{b}(t)$ is a normalized gaussian white noise such that $\mathbf{\Phi}_{bb}(\mathrm{s}) = \mathbf{1}_q$.

Exercise 1.3 *Let us consider again example 1.1 on the 1/4 vehicle model. Some experiments in a real environment have shown the PSD of the rolling noise for a vehicle at* $90\,km/h$ *on a secondary road could be approximated by:*

$$\Phi_{ww}(\omega) = \frac{\omega^2 + 10^4}{\omega^4 - 1.75\,10^4\,\omega^2 + 10^8}$$

[9] That is: a spectrum with only exponents of s^2; that is: the correspondent auto-correlation function $\boldsymbol{\phi}_{\mathbf{ww}}(\tau)$ is pair.

a) Give the spectrum in the s*-plane for* $w(t)$*.*

b) Give a MARKOV *representation of* $w(t)$*.*

c) Compute the variance of $w(t)$ *(the signal is assumed to be centered).*

Solution :

a) By definition: $\Phi_{ww}(\mathrm{s}) = \frac{-\mathrm{s}^2+10^4}{\mathrm{s}^4+1.75\,10^4\,\mathrm{s}^2+10^8}$.

b) A decomposition $\Phi_{ww}(\mathrm{s}) = G(-\mathrm{s})G(\mathrm{s})$ *is:*

$$\Phi_{ww}(\mathrm{s}) = \left(\frac{10^2+\mathrm{s}}{\mathrm{s}^2-50\mathrm{s}+10^4}\right)\left(\frac{10^2-\mathrm{s}}{s^2+50s+10^4}\right) .$$

All the stable poles of $\mathbf{\Phi_{ww}}(\mathrm{s})$ *must be gathered in* $G(\mathrm{s})$*, that allows the denominator of* $G(\mathrm{s})$ *to be determined but there is no condition on the stability of* $G(\mathrm{s})$ *zeros. So there are 2 different filters that provide an output signal with the spectrum* $\mathbf{\Phi_{ww}}(\mathrm{s})$ *when the input is a normalized white noise:*

$$G(\mathrm{s}) = \frac{10^2-\mathrm{s}}{\mathrm{s}^2+50\mathrm{s}+10^4} \quad \textit{and} \quad G'(\mathrm{s}) = \frac{10^2+\mathrm{s}}{\mathrm{s}^2+50\mathrm{s}+10^4}$$

If $G(\mathrm{s})$ *is chosen then a* MARKOV *representation of* $\mathbf{w}(t)$ *is (an horizontal companion form is proposed, for instance):*

$$\left\{\begin{array}{l} \dot{\mathbf{x}}_G(t) = \begin{bmatrix} 0 & 1 \\ -10^4 & -50 \end{bmatrix}\mathbf{x}_G(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} b(t) \\ w(t) = \begin{bmatrix} 10^2 & -1 \end{bmatrix}\mathbf{x}_G(t) \end{array}\right. . \tag{1.15}$$

c) The variance can be computed in two different ways:

- *from the spectrum in the* s*-plane and the initial value theorem (see remark 1.2):*

$$\Phi_{ww}(\mathrm{s}) = \frac{1/50\,\mathrm{s}+1/2}{\mathrm{s}^2+50\mathrm{s}+10^4} + \frac{-1/50\,\mathrm{s}+1/2}{\mathrm{s}^2-50\mathrm{s}+10^4} = \Phi^+_{ww}(\mathrm{s}) + \Phi^-_{ww}(-\mathrm{s})$$

$$\mathrm{var}_w = \phi_{ww}(\tau)|_{\tau=0} = \lim_{\mathrm{s}\to\infty} \mathrm{s}\Phi^+_{ww}(\mathrm{s}) = 1/50.$$

- *from the* MARKOV *model and the theorem 1.1: during steady-state, the covariance matrix* P *of the state vector* x_G *is the solution of the* LYAPUNOV *equation:*

$$\begin{bmatrix} 0 & 1 \\ -10^4 & -50 \end{bmatrix}\mathbf{P} + \mathbf{P}\begin{bmatrix} 0 & -10^4 \\ 1 & -50 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{P} = \begin{bmatrix} 10^{-6} & 0 \\ 0 & 10^{-2} \end{bmatrix}$$

$$\Rightarrow \text{var}_w = \sigma_w^2 = [10^2 \quad -1] \begin{bmatrix} 10^{-6} & 0 \\ 0 & 10^{-2} \end{bmatrix} \begin{bmatrix} 10^2 \\ -1 \end{bmatrix} = 1/50 \; .$$

□

1.4 Matlab illustration

The file `bruit.m`, given below, allows to illustrate the way to use Matlab macro-functions to solve exercise 1.3. This file also shows how to simulate, on a numerical computer, such a colored noise from a pseudo-white noise generated with a sample period dt. The Simulink file `simule_bruit.mdl` depicted in Figure 1.10 [10] can be simulated. Responses of pseudo-white noise $b(t)$ and rolling noise $w(t)$ are plotted in Figures 1.11 and 1.12, respectively. These responses highlight that the variance of $w(t)$ is independent of dt (since the sampling frequency $2\pi/dt$ is fast w.r.t the filter $G(s)$ dynamics) while the variance of $b(t)$ is equal to $1/dt$: one can see that this variance tends toward infinity if dt tend to zero as the variance of a continuous-time white noise is infinite (but the PSD of $b(t)$ is independent of dt and is equal to 1).

The example B.1 proposed in appendix completes this illustration by a discrete-time analysis (with a sampling period dt).

```
%============================================================
clear all close all
% Filter definition:
G=tf([-1 100],[1 50 10000])
% State space realization:
[A,B,C,D]=ssdata(G);
% Variance determination using Lyapunov equation:
P=lyap(A,B*B'); var_w=C*P*C'
%  ==> on can find the wanted result: variance=1/50.
%     (standard deviation: sigma=0.14)

%  Validation using a simulation: see SIMULINK file: simule_bruit.mdl

% Choice of a sampling period fast enough w.r.t. the filter dynamics (100 rd/s):
dt=0.0001;
% Simulation:
```

[10] Feel free to send an e-mail to `alazard@isae.fr` with "Introduction Kalman" for the subject if you want a copy of the 2 files `bruit.m` and `simule_bruit.mdl`.

```
sim('simule_bruit');
% Results plotting:
plot(b.time,b.signals.values,'k');
% Numerical computation of the variance of b(t):
var(b.signals.values) % One can find again: var_b=1/dt=10000.
figure
plot(w.time,w.signals.values,'k')
% Numerical computation of variance of w(t):
var(w.signals.values) % One can find: var_w=1/50 (approximately).

% The sampling period is now multiplied by a factor 10:
dt=0.001;
sim('simule_bruit');
figure(1) hold on
plot(b.time,b.signals.values,'g-');
var(b.signals.values) % One can find again: var_b=1/dt=1000.
figure(2) hold on
plot(w.time,w.signals.values,'g-')
var(w.signals.values) %  One can find: var_w=1/50 (approximately).
%=============================================================
```

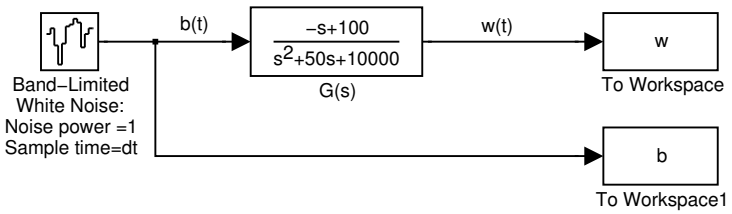

Figure 1.10: SIMULINK file `simule_bruit.mdl` for the rolling noise simulation.

1.5 Conclusion

Mathematical tools used to analyse continuous-time random signals and their transmission in linear systems were presented in this chapter. The notion of gaussian white noise was also introduced (this assumption is required in the Kalman filter developed in next chapter). The interest of such gaussian (centered) noises lies in the fact that they are entirely characterized by their autocorrelation function (or their spectrums in the s-plane if they are stationary).

The reader will find in appendix B some additional developments for the analysis of discrete-time random signals.

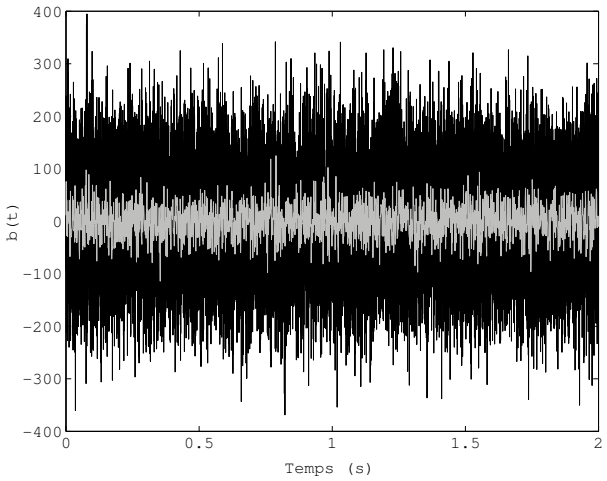

Figure 1.11: Responses of normalized white-noise $b(t)$ (with $dt = 0.0001\,s$: black; with $dt = 0.001\,s$: grey).

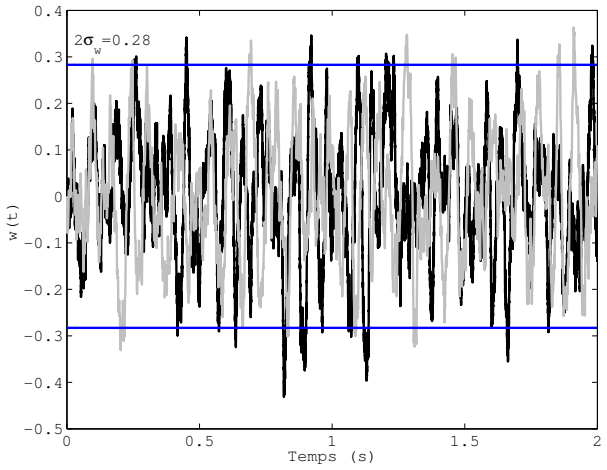

Figure 1.12: Responses of rolling noise $w(t)$ (with $dt = 0.0001\,s$: black; with $dt = 0.001\,s$: grey).

Chapter 2

Kalman filter

2.1 Principle of Kalman filtering

2.1.1 Kalman model

We consider again the model presented at the beginning of chapter 1. This model involves deterministic inputs $\mathbf{u}(t)$ and random inputs $\mathbf{w}(t)$ and $\mathbf{v}(t)$. So we will assume that the model of the perturbed plant can be always described by the following state space realization, also called **Kalman model**:

$$\left\{ \begin{array}{l} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{M}\mathbf{w}(t) \quad \text{state equation,} \quad \mathbf{x} \in \mathbb{R}^n,\ \mathbf{u} \in \mathbb{R}^m,\ \mathbf{w} \in \mathbb{R}^q \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{v}(t) \quad \text{measurement equation,} \quad \mathbf{y} \in \mathbb{R}^p,\ \mathbf{v} \in \mathbb{R}^p \end{array} \right. . \tag{2.1}$$

The objective of KALMAN filtering is to estimate the plant state $\mathbf{x}(t)$. This estimate, commonly noted $\widehat{\mathbf{x}}(t)$, is the output of the KALMAN filter.

The **Kalman model** (2.1) suggests an input-output oriented system between deteministic input $\mathbf{u}$ and output $\mathbf{y}$. In the field of signal processing, this input-output relationship can be quite different than the one used in servo-loop theory in that sense that $\mathbf{u}$ is not necessary the system control signal. Indeed, and more particularly in the design of navigation filter for aerospace vehicles, the **Kalman model** (2.1) can be used to describe the relationship between two kinds of measurements: primary measurements (gathered in vector $\mathbf{u}$) and secondary mesurements (gathered in vector $\mathbf{y}$). Then, the objective of the KALMAN filter is to perform the "best" measurement fusion between primary and secondary measurements.

Example 2.1 *Let us consider the motion of an aerospace vehicle (with a mass m) along a single translation degree-of-freedom. Its position along this degree-of-freedom is named $p(t)$. To control the postion $p(t)$, the spacecraft is fitted with an actuator (thruster). This actuator can be considered itself as a dynamic sub-system with a transfer function $A(\mathrm{s})$ between its input $u_c(t)$ (in fact the output of the control*

algorithm) and the output $F(t)$: the real thrust applied to the spacecraft (N). The spacecraft is instrumented with:

- *a linear accelorometer which measures the acceleration $\ddot{p}(t) = \frac{d^2p(t)}{dt^2}$ with a noise $w(t)$, that is: $\ddot{p}_m(t) = \ddot{p}(t) + w(t)$,*
- *a position sensor (GPS) which measures the position $p(t)$ with a noise $v(t)$, that is: $p_m(t) = p(t) + v(t)$.*

The control algorithm (out of the scope of this document) requires estimates of the position $\widehat{p}(t)$ and the velocity $\widehat{v}(t) = \dot{\widehat{p}}(t)$ of the spacecraft in order to compute the control signal $u_c(t)$ to be sent to the actuator. Note that the velocity is not measured. To solve such a problem, one can use a KALMAN *filter involving the accelerometer as primary measurement $u(t) = \ddot{p}_m(t)$ (seen as the input of the* KALMAN *model) and the position sensor as secondary mesurement $y(t) = p_m(t)$ (seen as the output of the* KALMAN *model). Obviously, the* **Kalman *model*** *between u (acceleration) and y (position) is a double integration and reads:*

$$\begin{aligned} \dot{p}(t) &= v(t) \\ \dot{v}(t) &= \ddot{p}(t) = u(t) - w(t) \\ y(t) &= p(t) + v(t) \end{aligned}$$

or using the general state-space representation (2.1) with $\mathbf{x}(t) = [p(t) \quad v(t)]^T$:

$$\left\{ \begin{array}{l} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} w(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) + 0u(t) + v(t) \end{array} \right. . \tag{2.2}$$

The block-diagram sketch of the whole system is represented in Figure 2.1. Of course such a KALMAN *filter cannot be used to estimate the internal state of the actuator sub-system $A(s)$ as the model of the actuator is not taken into account in the* KALMAN *model.*

□

2.1.2 Assumptions

The **Kalman model** (2.1) must meet the following assumptions:

H1: the (**A**, **C**) pair is detectable, i.e. there is no unstable and unobservable eigenvalue in the model,

H2: signals $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are **centered gaussian white noise** with **Power Spectral Densities (PSD) W** and **V** respectively, that is [1]:

[1] $\delta(\tau)$ is the Dirac distribution expressed in s^{-1}, τ is expressed in s.

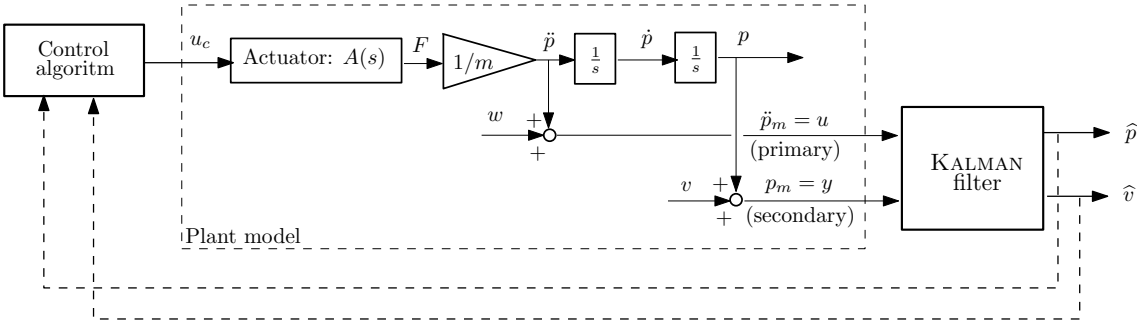

Figure 2.1: Block-diagram sketch for the control of an aerospace vehicle using a Kalman filter to hybridize primary (acceleration) and secondary (position) measurements.

- $\mathrm{E}[\mathbf{w}(t)\,\mathbf{w}(t+\tau)^T] = \mathbf{W}\,\delta(\tau)$,
- $\mathrm{E}[\mathbf{v}(t)\,\mathbf{v}(t+\tau)^T] = \mathbf{V}\,\delta(\tau)$
- $\mathrm{E}[\mathbf{w}(t)\,\mathbf{v}(t+\tau)^T] = \mathbf{0}$ (this last equality expresses the stochastic independence of noises $\mathbf{w}(t)$ and $\mathbf{v}(t)$: this assumption is introduced for reason of clearness but is not required in the general formulation of Kalman filter: the reader will find in reference [5] this general formulation taking into account a correlation between state noise and measurement noise).

H3: $\mathbf{V}$ is invertible (i.e. there is as much independent noise sources as measurements or each measurement is perturbed by its own noise [2]).

Remarks:

- Although the whole theory of Kalman filter can be applied to the non-stationary (or time-variant) case (for the plant and the noises), we will assume in the following that the plant and noises are stationary (or time-invariant), i.e: matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{M}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{W}$ are $\mathbf{V}$ independent of time t.
- The mean of a random signal, which is also called a **bias**, is considered to be deterministic and must be substracted to noises $\mathbf{w}(t)$ or $\mathbf{v}(t)$ to meet the assumption **H2** ($\mathbf{w}$ and $\mathbf{v}$ are **centered** noises). For instance, if random signal $\mathbf{w}(t)$ in state equation (2.1) is biased and if this bias $\mathrm{E}[\mathbf{w}(t)]$ is known, then the Kalman filter will be designed from the following model:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + [\mathbf{B} \quad \mathbf{M}] \begin{bmatrix} \mathbf{u}(t) \\ \mathrm{E}[\mathbf{w}(t)] \end{bmatrix} + \mathbf{M}(\mathbf{w}(t) - \mathrm{E}[\mathbf{w}(t)]) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{v}(t) \end{cases} .$$

[2] $\mathbf{V}$ is a positive definite matrix and $\mathbf{W}$ is a semi-definite positive matrix.

The new state noise $\bar{\mathbf{w}}(t) = \mathbf{w}(t) - \mathrm{E}[\mathbf{w}(t)]$ is now centered. If the bias $\mathrm{E}[\mathbf{w}(t)]$ is unknown then it can be modelled as an initial condition on a new integral state and the KALMAN filter will allow this bias to be estimated (see example in section 2.3).

- If noises are colored and characterized by spectrums in the s-plane then results of section 1.3.2 will allow the "color" (or frequency response) of the noises to be taken into account with a KALMAN model augmented by the **Markov representation** of these noises. For instance: if the spectrum in the s-plane $\mathbf{\Phi_{ww}}(\mathrm{s})$ of the centered random signal $\mathbf{w}(t)$ in (2.1) is known, then the decomposition $\mathbf{\Phi_{ww}}(\mathrm{s}) = \mathbf{G}(-\mathrm{s})\mathbf{G}^T(\mathrm{s})$ will allow a **Markov representation** of $\mathbf{w}(t)$ to be determined, that is: a state space realization of $\mathbf{G}(\mathrm{s})$ (see example 1.3):

$$\begin{cases} \dot{\mathbf{x}}_\mathbf{G}(t) = \mathbf{A_G}\mathbf{x_G}(t) + \mathbf{B_G}\mathbf{b}(t) \\ \mathbf{w}(t) = \mathbf{C_G}\mathbf{x_G}(t) \end{cases}$$

  where $\mathbf{b}(t)$ is a centered random signal with a unitary spectrum in the s-plane $\mathbf{\Phi_{bb}}(\mathrm{s}) = \mathbf{1}_q$ (i.e. a normalised white noise).

  The augmented model:

$$\begin{cases} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{x}}_\mathbf{G}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{MC_G} \\ \mathbf{0} & \mathbf{A_G} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x_G}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{u}(t) + \begin{bmatrix} \mathbf{0} \\ \mathbf{B_G} \end{bmatrix} \mathbf{b}(t) \\ \mathbf{y}(t) = [\mathbf{C} \quad \mathbf{0}] \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x_G}(t) \end{bmatrix} \end{cases}$$

  meets now all the assumptions of the KALMAN model.

In fact all the deterministic information it is possible to know in the system must be gather in the model (i.e. $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$, $\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$ and matrix $\mathbf{M}$). All the random information must be gather in noises $\mathbf{w}(t)$ and $\mathbf{v}(t)$. The state noise $\mathbf{w_x} = \mathbf{Mw}$ represents all the external perturbations (wind in the case of an aircraft, road unevenness in the case of a car, ...) and/or also modelling errors or uncertainties (difference between the tangent linear model and the non-linear model, neglected dynamics, ...): $\mathbf{w_x}$ is an **upper bound** of all what makes that the state does not evolve exactly as the deterministic model predicts ($\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$).

2.1.3 Structure of an unbiased estimator

A KALMAN filter is a dynamic system with 2 (vector) inputs: the deterministic control signal $\mathbf{u}$ and the measurement $\mathbf{y}$, that is: all known signals of the plant. The state $\widehat{\mathbf{x}}$ (or the output) of this filter is an estimate of the state $\mathbf{x}$ of the plant.

Let:

$$\dot{\widehat{\mathbf{x}}}(t) = \mathbf{A}_f\widehat{\mathbf{x}}(t) + [\mathbf{B}_f \quad \mathbf{K}_f]\begin{bmatrix}\mathbf{u}(t)\\ \mathbf{y}(t)\end{bmatrix} \tag{2.3}$$

$$= \mathbf{A}_f\widehat{\mathbf{x}}(t) + \mathbf{B}_f\mathbf{u}(t) + \mathbf{K}_f\mathbf{y}(t) \tag{2.4}$$

be the state space realization of this filter. Of course this **filter must be initialized with** $\widehat{\mathbf{x}}(t_0)$: the estimate of the state of the plant at the initial time t_0.

Let us denote $\boldsymbol{\varepsilon}(t) = \mathbf{x}(t) - \widehat{\mathbf{x}}(t)$ the state estimation error and $\boldsymbol{\varepsilon}(t_0) = \mathbf{x}(t_0) - \widehat{\mathbf{x}}(t_0)$ the initialization error.

Substituting the equation (2.4) to the state equation of (2.1) and using the measurement equation, we can write:

$$\begin{aligned}\dot{\varepsilon} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{M}\mathbf{w} - \mathbf{A}_f\widehat{\mathbf{x}} - \mathbf{B}_f\mathbf{u} - \mathbf{K}_f(\mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \mathbf{v})\\ &= (\mathbf{A} - \mathbf{K}_f\mathbf{C})\mathbf{x} - \mathbf{A}_f\widehat{\mathbf{x}} + (\mathbf{B} - \mathbf{B}_f - \mathbf{K}_f\mathbf{D})\mathbf{u} + \mathbf{M}\mathbf{w} - \mathbf{K}_f\mathbf{v}\\ &= (\mathbf{A} - \mathbf{K}_f\mathbf{C})\boldsymbol{\varepsilon} + (\mathbf{A} - \mathbf{K}_f\mathbf{C} - \mathbf{A}_f)\widehat{\mathbf{x}} + (\mathbf{B} - \mathbf{K}_f\mathbf{D} - \mathbf{B}_f)\mathbf{u} + \mathbf{M}\mathbf{w} - \mathbf{K}_f\mathbf{v}\,. \end{aligned} \tag{2.5}$$

As noises $\mathbf{w}$ and $\mathbf{v}$ are gaussian and as the system is linear, on can state that $\boldsymbol{\varepsilon}(t)$ is a gaussian random signal. We are going to characterize the expected value of $\boldsymbol{\varepsilon}(t)$.

Unbiased estimator: first of all, we wish the estimator to be unbiased, that is:

- whatever the input profile $\mathbf{u}(\tau)$ applied in the time interval $\tau \in [t_0,\ t]$,
- whatever the initialization $\widehat{\mathbf{x}}(t_0)$,

we wish the estimation error expected value tends towards 0 when t tends towards infinity.

As noises $\mathbf{w}$ and $\mathbf{v}$ are centered, we can write:

$$\dot{\mathrm{E}}[\boldsymbol{\varepsilon}(t)] = \mathrm{E}[\dot{\boldsymbol{\varepsilon}}(t)] = (\mathbf{A}-\mathbf{K}_f\mathbf{C})\mathrm{E}[\boldsymbol{\varepsilon}(t)]+(\mathbf{A}-\mathbf{K}_f\mathbf{C}-\mathbf{A}_f)\mathrm{E}[\widehat{\mathbf{x}}(t)]+(\mathbf{B}-\mathbf{K}_f\mathbf{D}-\mathbf{B}_f)\mathbf{u}(t)$$

Then $\lim_{t\to\infty}\mathrm{E}[\boldsymbol{\varepsilon}(t)] = \mathbf{0}$, $\forall\ \mathbf{u}(t)$, $\forall\ \mathrm{E}[\widehat{\mathbf{x}}(t)]$, if and only if:

$$\mathbf{A}_f = \mathbf{A} - \mathbf{K}_f\mathbf{C}, \quad \mathbf{B}_f = \mathbf{B} - \mathbf{K}_f\mathbf{D} \tag{2.6}$$

$$\text{and} \quad \mathbf{A} - \mathbf{K}_f\mathbf{C} \quad \text{is stable.} \tag{2.7}$$

Indeed, from theorem 1.1 (page 21):

$$\mathrm{E}[\boldsymbol{\varepsilon}(t)] = \mathrm{e}^{(\mathbf{A}-\mathbf{K}_f\mathbf{C})(t-t_0)}\boldsymbol{\varepsilon}(t_0) \quad \text{and} \quad \lim_{t\to\infty}\mathrm{E}[\boldsymbol{\varepsilon}(t)] = 0\,.$$

If (2.6) is used in (2.4), the Kalman filter realization becomes:

$$\boxed{\dot{\widehat{\mathbf{x}}} = (\mathbf{A}\widehat{\mathbf{x}} + \mathbf{B}\mathbf{u}) + \mathbf{K}_f\,(\mathbf{y} - \mathbf{C}\widehat{\mathbf{x}} - \mathbf{D}\mathbf{u})}\,. \tag{2.8}$$

We recognize, in the first term of the right-hand member of equation (2.8), the deterministic model of the plant ($\mathbf{A}\widehat{\mathbf{x}}+\mathbf{B}\mathbf{u}$). This model is used to predict the evolution of the plant state from the current estimation $\widehat{\mathbf{x}}$. This prediction is in fact an on-line simulation of the plant model. This model being wrong, the prediction is corrected (updated) with the difference between the measurement $\mathbf{y}$ and the prediction of the measurement $\widehat{\mathbf{y}}=\mathbf{C}\widehat{\mathbf{x}}+\mathbf{D}\mathbf{u}$ through the filter gain $\mathbf{K}_f$. This difference $\mathbf{y}-\widehat{\mathbf{y}}$ is also called the *innovation*. The block-diagram of such a filter is depicted in Figure 2.2. This **structure** ensures that the estimator is unbiased whatever the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ of the plant and the gain $\mathbf{K}_f$ such that $\mathbf{A}-\mathbf{K}_f\mathbf{C}$ is stable (that justifies the assumption **H1**: the existence of an unstable and unobservable mode in the plant does not allow a stabilizing gain $\mathbf{K}_f$ to be determined and so an unbiased estimator to be built).

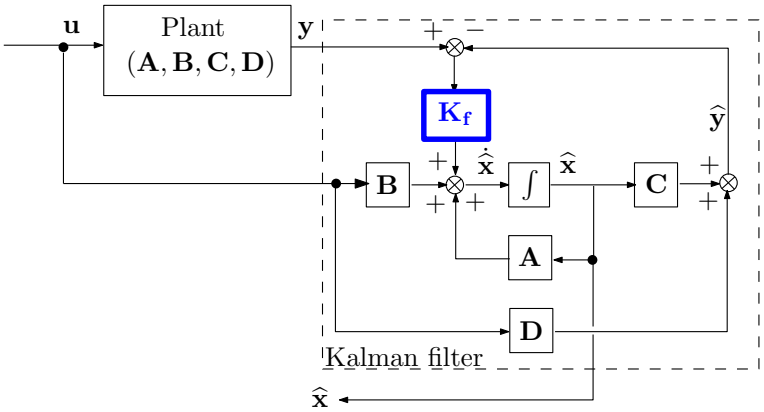

Figure 2.2: KALMAN filter block-diagram.

2.2 Minimal error variance estimator

The gain $\mathbf{K}_f$ is computed according to the **confidence we have in the model** (this confidence is expressed by the PSD $\mathbf{W}$: the lower this confidence is, the greater $\mathbf{W}$ must be) with respect to the **confidence we have in the measurement** (expressed by the PSD $\mathbf{V}$). If the model is very precise ($\mathbf{W}$ is very low) and if the measurement is very noisy ($\mathbf{V}$ is very great) then the gain $\mathbf{K}_f$ must be very low (the prediction is good). In fact, among all the stabilizing gains $\mathbf{K}_f$, we are going to choose the one that **minimizes the variance of the state estimation error** $\boldsymbol{\varepsilon}(t)$ $\forall t$ (in fact the one that minimizes simultaneously the variances for all the n state estimation error components). We recall (see previous section) that $\boldsymbol{\varepsilon}(t)=\mathbf{x}(t)-\widehat{\mathbf{x}}(t)$ is a multivariate (with n components), centered (unbiased), gaussian random signal. The gaussian feature of this centered signal allowes to state that if the variance of

the estimation error is really minimized then, $\widehat{\mathbf{x}}(t)$ is the best estimate of $x(t)$.

2.2.1 General solution

So we seek $\mathbf{K}_f$ minimizing:

$$J_i(t) = \mathrm{E}[\boldsymbol{\varepsilon}_i(t)^2], \quad \forall\, i = 1, \ldots, n \tag{2.9}$$

where $\boldsymbol{\varepsilon}_i(t)$ is the i-th component of $\boldsymbol{\varepsilon}(t)$.

Let us denote $\mathbf{P}(t) = \mathrm{E}[(\mathbf{x}(t) - \widehat{\mathbf{x}}(t))(\mathbf{x}(t) - \widehat{\mathbf{x}}(t))^T] = \mathrm{E}[\boldsymbol{\varepsilon}(t)\boldsymbol{\varepsilon}(t)^T]$ the $n \times n$ estimation error covariance matrix. Then:

$$J_i(t) = \mathbf{s}_i^T \mathbf{P}(t) \mathbf{s}_i = \mathbf{P}_{(i,i)}(t)$$

where the vector $\mathbf{s}_i = [0 \ldots 0,\ 1,\ 0 \ldots 0]^T$ (i.e. 0 everywhere except 1 on the i-th component) and $\mathbf{P}_{(i,i)}(t)$ is the i-th term of the diagonal of $\mathbf{P}(t)$.

Returning (2.6) in (2.5), the evolution of $\boldsymbol{\varepsilon}(t)$ is governed by the following state space equation:

$$\dot{\boldsymbol{\varepsilon}}(t) = (\mathbf{A} - \mathbf{K}_f\mathbf{C})\boldsymbol{\varepsilon}(t) + [\mathbf{M} \quad -\mathbf{K}_f] \begin{bmatrix} \mathbf{w}(t) \\ \mathbf{v}(t) \end{bmatrix}, \tag{2.10}$$

with:

$$\mathrm{E}\left[\begin{bmatrix} \mathbf{w}(t) \\ \mathbf{v}(t) \end{bmatrix} [\mathbf{w}^T(t+\tau) \quad \mathbf{v}^T(t+\tau)] \right] = \begin{bmatrix} \mathbf{W}_{q\times q} & \mathbf{0}_{q\times p} \\ \mathbf{0}_{p\times q} & \mathbf{V}_{p\times p} \end{bmatrix} \delta(\tau)\ .$$

The theorem 1.1 (page 21) can be applied directly and allows to conclude that the estimation error covariance $P(t)$ is governed by the differential Lyaponov equation:

$$\begin{aligned} \dot{\mathbf{P}}(t) &= (\mathbf{A} - \mathbf{K}_f\mathbf{C})\mathbf{P}(t) + \mathbf{P}(t)(\mathbf{A} - \mathbf{K}_f\mathbf{C})^T + [\mathbf{M} \quad -\mathbf{K}_f] \begin{bmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{0} & \mathbf{V} \end{bmatrix} \begin{bmatrix} \mathbf{M}^T \\ -\mathbf{K}_f^T \end{bmatrix} \\ &= (\mathbf{A} - \mathbf{K}_f\mathbf{C})\mathbf{P}(t) + \mathbf{P}(t)(\mathbf{A} - \mathbf{K}_f\mathbf{C})^T + \mathbf{M}\mathbf{W}\mathbf{M}^T + \mathbf{K}_f\mathbf{V}\mathbf{K}_f^T\ , \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \dot{J}_i(t) &= \mathbf{s}_i^T(\mathbf{A} - \mathbf{K}_f\mathbf{C})\mathbf{P}(t)\mathbf{s}_i + \mathbf{s}_i^T\mathbf{P}(t)(\mathbf{A} - \mathbf{K}_f\mathbf{C})^T\mathbf{s}_i + \mathbf{s}_i^T\mathbf{M}\mathbf{W}\mathbf{M}^T\mathbf{s}_i + \mathbf{s}_i^T\mathbf{K}_f\mathbf{V}\mathbf{K}_f^T\mathbf{s}_i \\ &= \mathbf{s}_i^T\mathbf{A}\mathbf{P}(t)\mathbf{s}_i - \mathbf{K}_{f(i,:)}(t)\mathbf{C}\mathbf{P}(t)\mathbf{s}_i + \mathbf{s}_i^T\mathbf{P}(t)\mathbf{A}^T\mathbf{s}_i - \mathbf{s}_i^T\mathbf{P}(t)\mathbf{C}^T\mathbf{K}_{f(i,:)}^T(t) \\ &\quad + \mathbf{s}_i^T\mathbf{M}\mathbf{W}\mathbf{M}^T\mathbf{s}_i + \mathbf{K}_{f(i,:)}(t)\mathbf{V}\mathbf{K}_{f(i,:)}^T(t) \end{aligned}$$

where $\mathbf{K}_{f(i,:)}(t)$ is the i-th row of the matrix $\mathbf{K}_f(t)$.

To minimize $J_i(t)$ (w.r.t. $\mathbf{K}_f$ and $\forall\ t > t_0$), it is sufficient to minimize $\dot{J}_i(t)$ w.r.t. $\mathbf{K}_{f(i,:)}(t)$ ($\forall\ t > t_0$). Since $\dot{J}_i(t)$ is quadratic in $K_{f(i,:)}(t)$, the first-order optimal condition is sufficient to find the global minimum:

$$\frac{\partial(\dot{J}_i(t))}{\partial \mathbf{K}_{f(i,:)}(t)} = -\mathbf{s}_i^T\mathbf{P}(t)\mathbf{C}^T - \mathbf{s}_i^T\mathbf{P}(t)\mathbf{C}^T + 2\mathbf{K}_{f(i,:)}(t)\mathbf{V} = 0$$

$$\Rightarrow \quad \mathbf{K}_{f(i,:)}(t) = \mathbf{s}_i^T \mathbf{P}(t)\mathbf{C}^T\mathbf{V}^{-1} \; . \tag{2.12}$$

Thus, the i-th row of $\mathbf{K}_f(t)$ is the i-th row of $\mathbf{P}(t)\mathbf{C}^T\mathbf{V}^{-1}$. For $i = 1, \ldots, n$, one can feed all the rows of $\mathbf{K}_f(t)$ and:

$$\Rightarrow \quad \boxed{\mathbf{K}_f(t) = \mathbf{P}(t)\mathbf{C}^T\mathbf{V}^{-1}} \; . \tag{2.13}$$

From(2.13) and (2.11), we get:

$$\boxed{\dot{\mathbf{P}}(t) = \mathbf{A}\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}^T - \mathbf{P}(t)\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}\mathbf{P}(t) + \mathbf{M}\mathbf{W}\mathbf{M}^T} \; . \tag{2.14}$$

This RICCATI differential equation must be integrated from t_0 to t and must be initialized with $\mathbf{P}(t_0)$ that expresses the **confidence we have in the filter initialization** $\widehat{\mathbf{x}}(t_0)$:

$$\mathbf{P}(t_0) = \mathrm{E}[(\mathbf{x}(t_0) - \widehat{\mathbf{x}}(t_0))(\mathbf{x}(t_0) - \widehat{\mathbf{x}}(t_0))^T] \; .$$

In appendix C, an analytical solution for the general matricial differential RICCATI equation (2.14) is proposed and illustrated on an academic example.

The gain $\mathbf{K}_f(t)$ is then computed from $\mathbf{P}(t)$ and equation (2.13). **The Kalman filter is thus a time-variant system**.

Equations (2.8), (2.13) and (2.14) constitute the continuous-time KALMAN filter equations and must be integrated from initialization $\widehat{\mathbf{x}}(t_0)$ and $\mathbf{P}(t_0)$. The integration of (2.14) and the computation of $\mathbf{K}_f(t)$ (2.13) can be performed in-line or off-line. In the last case, the time-law $\mathbf{K}_f(t)$ must be uploaded in the on-board computer. From a practical point of view, the implementation of the KALMAN filter will be performed in discrete-time on a numerical computer. The KALMAN filter state equation (2.8) can be discretized (integration using rectangular (EULER or trapezoidal (TUSTIN approximations, ..., if only the steady state behavior is considered). One can also prefer to directly design the KALMAN filter in discrete-time (see section 2.4.2). Lastly, all the filter equations are defined by the data of the problem, that is, the set of matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{M}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{W}$ and $\mathbf{V}$.

2.2.2 Kalman filter steady state

During the steady state (that is: once the transient response due to initialization error is finished), the estimation error becomes a stationary random signal (this is the case for all signals in the block diagram of Figure 2.2). So we can write:

$$\dot{\mathbf{P}}(t) = 0 \; .$$

$\mathbf{P}$ is a constant positive definite matrix (the covariance of the state estimation error in steady state) and is the only positive solution of the ALGEBRAIC RICCATI EQUATION:

$$\boxed{\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T - \mathbf{P}\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}\mathbf{P} + \mathbf{M}\mathbf{W}\mathbf{M}^T = \mathbf{0}} \tag{2.15}$$

The filter gain $\mathbf{K}_f = \mathbf{P}\mathbf{C}^T\mathbf{V}^{-1}$ becomes also stationary.

On can easily verify (from the steady state of LYAPUNOV equation (2.11)) that the positiveness of $\mathbf{P}$ implies the stability of the filter, that is all the eigenvalues of $\mathbf{A} - \mathbf{K}_f\mathbf{C}$ have a negative real part.

The reader will find in [1], a general method to compute the positive solution of an algebraic RICCATI equation. From a practical point of view, keep in mind that such solvers are available in most softwares for computer-based control design: macro-function `lqe` of **Matlab** or **Scilab** control packages. This function provides directly $\mathbf{P}$ and $\mathbf{K}_f$ from matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{M}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{W}$ and $\mathbf{V}$ (see also macro-function `care` and `kalman` under **Matlab**).

2.2.3 Kalman filter tuning

For a given model (matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{M}$, $\mathbf{C}$, $\mathbf{D}$), the filter gain $\mathbf{K}_f$ and its evolution as a function of time depend only on:

- $\mathbf{W}$: the "confidence" we have in the state equation,
- $\mathbf{V}$: the "confidence" we have in the measurement,
- $\mathbf{P}(t_0)$: the "confidence" we have in the initialization.

In steady state, if the system and the noises are stationary, the KALMAN gain $\mathbf{K}_f$ is constant and its value depends only on $\mathbf{W}$ and $\mathbf{V}$.

Furthermore, the gain $\mathbf{K}_f$ and the estimation error response $\boldsymbol{\varepsilon}(t)$ depend only on the relative weight of $\mathbf{P}(t_0)$ w.r.t. $\mathbf{V}$ (during the transient response) and the relative weight of $\mathbf{W}$ w.r.t. $\mathbf{V}$ (during the steady state). Indeed, it can be easy to check that the gain $\mathbf{K}_f$ (and so the filter) does not change if these 3 tuning matrices $\mathbf{W}$, $\mathbf{V}$ and $\mathbf{P}(t_0)$ are multiplied by a scaling factor α. But keep in mind that the estimation error covariance $\mathbf{P}$ will be multiplied by α. So the designer must be sure that these 3 matrices are realistic if he wants to use $\mathbf{P}(t)$ to analyse the estimation quality and conclude for instance: *the probability for the i-th component of the state $\mathbf{x}_i(t)$ to be between $\widehat{\mathbf{x}}_i(t) - 2\sqrt{\mathbf{P}_{(i,i)}(t)}$ and $\widehat{\mathbf{x}}_i(t) + 2\sqrt{\mathbf{P}_{(i,i)}(t)}$* is equal to 0.95 (gaussian variable property). From a practical point of point, the KALMAN filter must be validate on a **validation model** which takes into account realistic measurement noises and a modelling, as accurate as possible, of all the disturbances (non-linearities, external perturbations,...) that have been overvalued by a state noise $\mathbf{w}(t)$ in the **Kalman model** also called **synthesis model**. Such a **validation model** cannot be used as **synthesis model** for 2 reasons: first, it does not generally fulfill the assumptions **H1**, **H2**, **H3** and secondly it will lead to a too much complex KALMAN filter from the implementation point of view.

Whatever the tuning, one can check that the KALMAN estimation error covariance $\mathbf{P}(t)$ is always lower than the covariance propagated in the state equation (that is without using measurement and governed by $\dot{\mathbf{P}} = \mathbf{AP} + \mathbf{PA}^T + \mathbf{MWM}^T$). In the same way, the estimation error covariance of the output $\mathbf{y}_p$ (without noise: $\mathbf{y}_p = \mathbf{Cx} + \mathbf{Du}$), that is: $\mathbf{CP}(t)\mathbf{C}^T$, is always lower that the measurement noise covariance (which is infinite in the continuous-time case !! but this property is also true in the discrete-time case where the measurement noise covariance is finite, see section 2.4). So it is always better to use $\widehat{\mathbf{y}}_p = \mathbf{C}\widehat{\mathbf{x}} + \mathbf{Du}$ rather than the direct measurement $\mathbf{y}$ to estimate the real output $\mathbf{y}_p$ of the plant.

The designer could use the following trade-off to tune qualitatively the estimation error time-response.

Influence of $\mathbf{P}(t_0)$ ./. $\mathbf{V}$: during the transient response, the initial estimation error $\boldsymbol{\varepsilon}_0$ will be corrected as faster (and the KALMAN gain will be as greater) as $\mathbf{P}(t_0)$ is greater w.r.t. $\mathbf{V}$. But the estimate will be affected by the measurement noise because we have a good confidence in this measurement.

Influence of $\mathbf{W}$ ./. $\mathbf{V}$: during the steady state, the gain $\mathbf{K}_f$ will be as lower, and the estimation response will be as smoother, as $\mathbf{W}$ is weaker w.r.t. $\mathbf{V}$ (we have confidence in the model). But, if the plant is subject to a perturbation which has been under-estimated by this too weak choice of $\mathbf{W}$, the estimate $\widehat{\mathbf{x}}$ will not track the real state $\mathbf{x}$ or will have a too long updating time. Furthermore the estimation error covariance $\mathbf{P}$ will not be representative of this phenomena.

These behaviors will be illustrated in the following example. Although this is a first order example, it allows to highlight general trade-offs that can be encountered on more complex applications (continuous or discrete-time).

2.3 Corrected exercises

2.3.1 First order system

Statement: let us consider a stable first order system

$$G(\mathrm{s}) = \frac{1}{\mathrm{s} - a} \quad \text{with} \quad a < 0 \,.$$

The measurement y of the system output x is perturbed by a white noise $v(t)$ with a unitary PSD ($V = 1$). The input of this system is subject to an unknown and variable transmission delay (the maximal value of this delay is denoted T). For the design of the KALMAN filter we want to perform on this system, this perturbation (unknown delay) will be roughly modelled by a white noise $w(t)$, with a PSD W, independent of v, acting directly on the input signal according to Figure 2.3.

Questions:

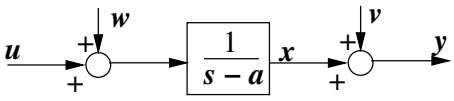

Figure 2.3: KALMAN synthesis model.

a) Set up the KALMAN model for this problem and compute the KALMAN filter, to estimate the state (or output) x of the plant. Give results as a function of a, W and P_0 (the initial estimation error variance). Study the asymptotic behavior of the filter gain K_f when W tends towards 0.

b) Build a simulator using *Matlab/Simulink* (involving a validation model with a real transmission delay T) in order to analyse the influence of parameters W and P_0 on the estimation error response $\varepsilon(t) = x(t) - \widehat{x}(t)$ and on its variance $P(t)$.
Numerical application:

- $a = -1$, $x_0 = 20$, $\widehat{x}_0 = 0$,
- $T = 0, \quad 0.1, \quad 0.5, \quad 1$(s) (these various values will be tested),
- $u(t)$: square signal with a magnitude of 10 and a period of $5\,s$,
- $dt = 0.01\,s$ (integration step for the simulator).

c) What do you propose to improve the estimation quality in the case of an important and known delay T (for instance: $T = 1\,s$) ?

Correction: Question a): the KALMAN model is:

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & ax(t) + u(t) + w(t) \\ y(t) & = & x(t) + v(t) \end{array} \right. \qquad (n = 1; m = 1; p = 1),$$

with $\mathrm{E}[w(t)w(t+\tau)] = W\delta(\tau)$, $\mathrm{E}[v(t)v(t+\tau)] = 1\,\delta(\tau)$ and $\mathrm{E}[v(t)v(t+\tau)] = 0$.

The first step in the determination of the KALMAN filter consists in solving the differential RICCATI equation (2.14) which is a scalar equation in this case:

$$\dot{P}(t) = 2aP(t) - P(t)^2 + W \ . \qquad (2.16)$$

In appendix C, an analytical solution for the general matricial differential RICCATI equation is proposed and illustrated (see section C.2) on this example. The reader will also find in ([9]) a general method to solve such a differential equation in the scalar case. This method consists in:

- looking for a constant particular solution (because the coefficients of the differential equation do not depend on time t). This solution corresponds in fact to steady-state solution $p = p_\infty$ which must solve:

$$P_\infty^2 - 2aP_\infty - W = 0 \; .$$

The only positive solution is $\boxed{P_\infty = a + \sqrt{a^2 + W}}$,

- looking for a general solution by quadrature, that is a solution of the form:

$$P(t) = P_\infty + \frac{1}{z(t)} \; .$$

The expansion of $\dot{P}(t)$ leads to:

$$\dot{z}(t) = 2\sqrt{a^2 + W}\, z(t) + 1 \; .$$

The integration of this differential equation using the varying constant method gives:

$$z(t) = \frac{1}{2\sqrt{a^2 + W}} \left[\mathrm{e}^{2\sqrt{a^2+W}(t-t_0)} - 1 \right] + \mathrm{e}^{2\sqrt{a^2+W}(t-t_0)} z_0$$

where t_0 is the initial time and $z_0 = z(t_0)$ is the initial condition on $z(t)$:

$$z_0 = \frac{1}{P_0 - P_\infty}$$

where P_0 is the initial condition on $P(t)$. Let us denote $k = 2\sqrt{a^2 + W}$, then:

$$\begin{aligned} P(t) &= P_\infty + \frac{k(P_0 - P_\infty)}{(P_0 - P_\infty)(\mathrm{e}^{k(t-t_0)} - 1) + k\mathrm{e}^{k(t-t_0)}} \\ &= P_\infty + \frac{k(P_0 - P_\infty)}{\mathrm{e}^{k(t-t_0)}(P_0 - P_\infty + k) + P_\infty - P_0} \; . \end{aligned} \tag{2.17}$$

Lastly: $K_f(t) = P(t)$.

If $W = 0$ then $P_\infty = a + |a| = 0$ if $a < 0$ (stable system)

$$\Rightarrow \quad \lim_{t \to \infty} K_f = 0 \; .$$

Once the initial error is corrected, only the model is used to estimate x. This is logic because the model is assumed to be perfect when W is set to 0. That property holds only if the system is stable. Indeed, if the system is unstable ($a > 0$) then:

$$\Rightarrow \quad \lim_{t \to \infty} K_f = 2a \; .$$

This is the value of the gain which allows the filter dynamics to be stable while minimizing the effect of the measurement noise on the estimation error variance.

Question b): the Matlab/Simulink simulator is depicted in Figure 2.4. The various parameters used in this Simulink file (`simuKalman.mdl`) are written in this Figure. The Matlab function `Kf_t.m` used to implement the time-variant gain K_f is given in appendix D with the script-file `demoKalman.m` to declare the various parameters and to plot results [3]. Different simulation results are presented in Figures 2.5 to 2.9 :

- Figure 2.5: The confidence P_0 in the initialization is not at all representative of the initial estimation error which is in fact very important ($\varepsilon_0 = x_0 - \widehat{x}_0 = 20$). The transient response to correct this initial error is thus long and the estimation of x $\pm 2\sigma$ (with $\sigma(t) = \sqrt{P(t)}$) does not allow to frame the real value of x during this transient response.

- Figure 2.6: If this confidence in the initialisation is degraded ($P_0 = 100$), this transient response becomes more faster. The measurement is now more exploited by the filter. The estimate is therefore a little bit noisy during this transient response. In steady state, this estimate becomes smoother because we are confident in the model (W is low). One can note that the estimate is unbiased. Lastly the estimation is good because in this simulation, the validation model (with $T = 0$) coincides exactly with the Kalman synthesis model.

- Figure 2.7: If now a $1\,s$ delay is taken into account in the validation model, the (good) confidence we have in the synthesis model ($W = 1$) does not allow to be representative of real model errors. The filter is confident in a wrong model and does not use the measurement enough: this filter has a very long reaction time.

- Figure 2.8: If now it is specified that the model is not so good ($W = 100$), the filter is more confident with measurements: the estimate is more sensitive to the measurement noise but have a good reaction time.

- Figure 2.9 (answer to question c)): if now the delay is known ($T = 1\,s$), it is possible to take this delay into account in the Kalman synthesis model in two different ways: firstly in propagating this delay in the prediction equation or secondly (what it is proposed here: see file `demoKalman.m`) in using a Pade filter in series with the model.

[3] Thanks to send an email to `alazard@supaero.fr` with "Introduction Kalman"" for the subject if you wish a copy of these files.

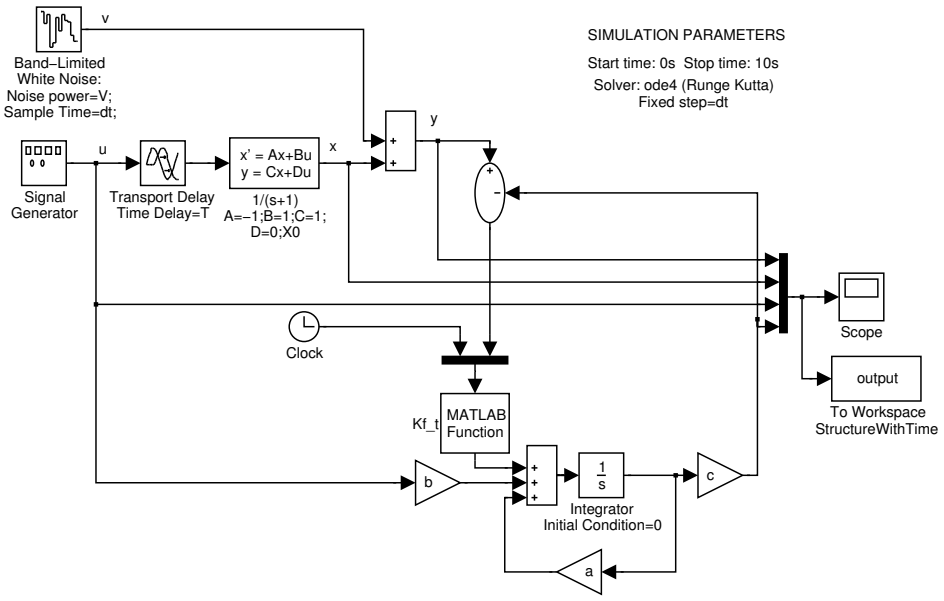

Figure 2.4: SIMULINK file `simuKalman.mdl` for the KALMAN filter simulation.

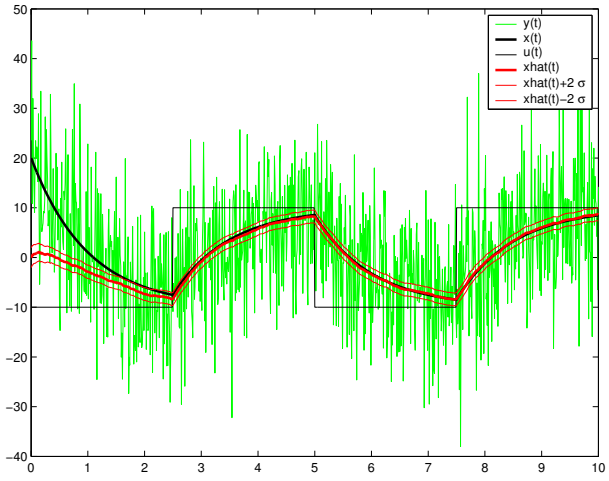

Figure 2.5: Simulation with $P_0 = 1$, $W = 1$, $T = 0$.

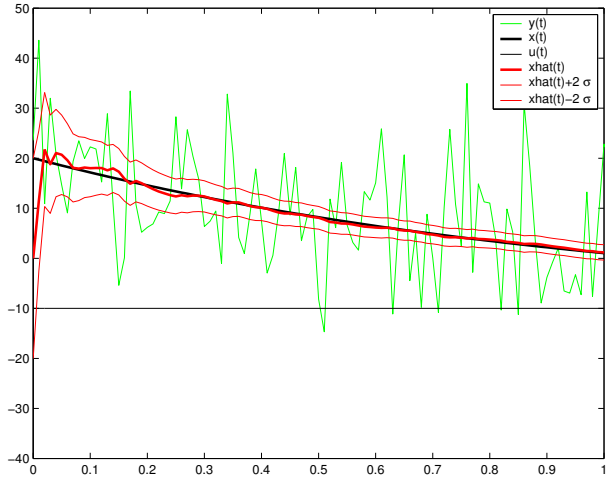

Figure 2.6: Simulation with $P_0 = 100$, $W = 1$, $T = 0$ (zoom around the transient response) .

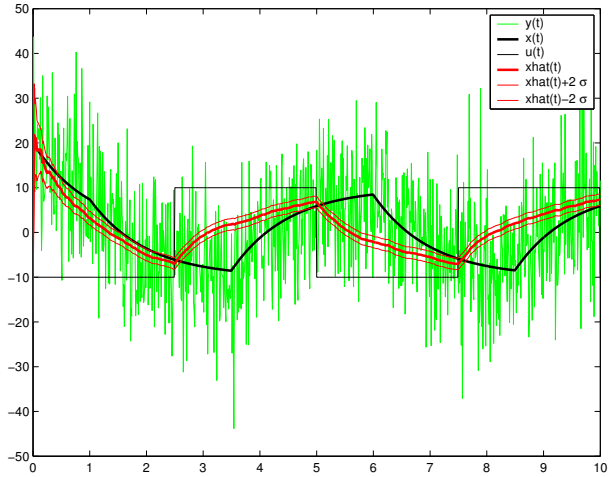

Figure 2.7: Simulation with $P_0 = 100$, $W = 1$, $T = 1$.

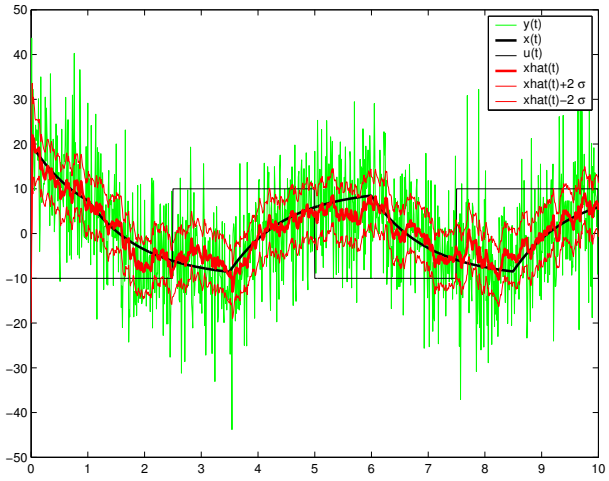

Figure 2.8: Simulation with $P_0 = 100$, $W = 100$, $T = 1$.

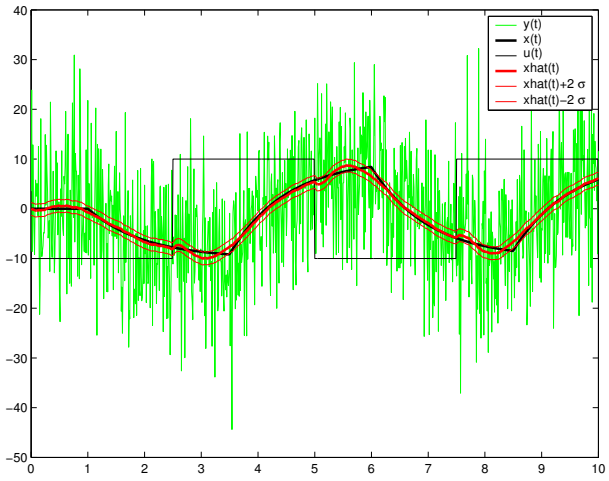

Figure 2.9: Simulation of the stationary KALMAN filter designed on a model taking into account a second order PADE approximation of the delay ($W = 1$, $T = 1\,s$).

2.3.2 Bias estimation

Statement: a vehicle is moving along an Ox axis. The velocity $\dot{x}$ and the position x of this vehicle are measured. These measures are denoted v_m and x_m, respectively. The measurement x_m is perturbed by a centered gaussian white noise $v(t)$ with a unitary PSD $V = 1\,(m^2/Hz)$: $x_m(t) = x(t) + v(t)$.
The measurement v_m is biased by a signal $b(t)$ which can be modelled as a step function with a unknown magnitude b_0: $v_m(t) = \dot{x}(t) + b(t)$.
From these 2 measurements v_m et x_m, we want to built a Kalman filter to estimate the position $x(t)$ and the bias $b(t)$.

1) Find the state space equations of the Kalman model with x and b as state variables, v_m as input variable, x_m as output variable.

2) Draw a block diagram representation of this model.

In fact the bias $b(t)$ can derive with time. To take into account these variations, it is assumed that the time-derivative of the bias is perturbed by a white noise $w(t)$ with a PSD $W = q^2$ (independent of $v(t)$):

$$\dot{b}(t) = w(t)\ .$$

3) Give the new Kalman model equation, compute the steady state Kalman gain (as a function of q) and give the state space representation of the filter allowing estimates $\widehat{x}$ and $\widehat{b}$ to be computed from x_m and v_m.

4) How would you proceed to estimate the velocity of the vehicle $\widehat{\dot{x}}$?

5) Compute the transfer matrix $\mathbf{F}(\mathrm{s})$ of the filter:

$$\begin{bmatrix} \widehat{X}(\mathrm{s}) \\ \widehat{\dot{X}}(\mathrm{s}) \end{bmatrix} = \mathbf{F}(\mathrm{s}) \begin{bmatrix} X_m(\mathrm{s}) \\ V_m(\mathrm{s}) \end{bmatrix}\ .$$

6) Comment this transfer (particularly $\widehat{\dot{X}}(\mathrm{s})/V_m(\mathrm{s})$) as a function of q. Demonstrate that this filter $\mathbf{F}(\mathrm{s})$ provides perfect estimates if the measurements are perfect.

Correction: **Question 1.** On can directly state:

$$\begin{cases} v_m(t) &= \dot{x}(t) + b(t) \\ x_m(t) &= x(t) + v(t) \end{cases} \quad \text{so:} \quad \begin{cases} \dot{x} &= -b + v_m \\ x_m &= x + v \end{cases}\ .$$

The bias is modeled as an integral effect ($\dot{b} = 0$) with an unknown initial condition b_0. Thus we have:

$$\left\{ \begin{array}{rcl} \begin{bmatrix} \dot{x} \\ \dot{b} \end{bmatrix} & = & \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ b \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_m \\ x_m & = & \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ b \end{bmatrix} + v \end{array} \right. \tag{2.18}$$

with $\mathrm{E}[v(t)v^T(t+\tau)] = 1\,\delta(\tau)$.

Question 2. The block diagram of the KALMAN model is depicted in Figure 2.10

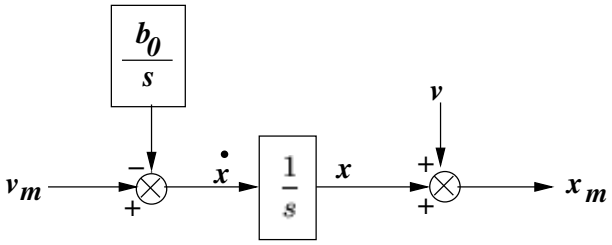

Figure 2.10: Synthesis model block-diagram.

Question 3. If the KALMAN filter is designed on model 2.18, we will find a null gain in steady state because the state equation is not perturbed by a noise: once the bias is estimated, the filter will not be able to detect an eventual drift of this parameter. To overcome this problem and to take into account the bias is not constant, a noise w is introduced on $\dot{b}(t)$:

$$\left\{ \begin{array}{rcl} \begin{bmatrix} \dot{x} \\ \dot{b} \end{bmatrix} & = & \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ b \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_m + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \\ x_m & = & \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ b \end{bmatrix} + v \end{array} \right. \tag{2.19}$$

with $\mathrm{E}[w(t)w^T(t+\tau)] = q^2\delta(\tau)$. This model is in the form of (2.1) and matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{M}$ can be identified.

The steady state solution reads: $\mathbf{K}_f = \mathbf{P}\mathbf{C}^T\mathbf{V}^{-1}$ with $\mathbf{P} = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}$ **positive** solution of the algebraic RICCATI equation:

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} + \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & q^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So:

$$\left\{ \begin{array}{r} 2p_{12} + p_1^2 = 0 \\ p_2 + p_1 p_{12} = 0 \\ p_{12}^2 = q^2 \end{array} \right. .$$

The positive solution is:

$$\mathbf{P} = \begin{bmatrix} \sqrt{2q} & -q \\ -q & q\sqrt{2q} \end{bmatrix} \quad \Rightarrow \quad \boxed{\mathbf{K}_f = \begin{bmatrix} \sqrt{2q} \\ -q \end{bmatrix}}.$$

The filter state space equation is (from (2.8)):

$$\begin{bmatrix} \dot{\widehat{x}} \\ \dot{\widehat{b}} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \widehat{x} \\ \widehat{b} \end{bmatrix} + \mathbf{B}v_m + \mathbf{K}_f \left(x_m - \mathbf{C} \begin{bmatrix} \widehat{x} \\ \widehat{b} \end{bmatrix} \right) = (\mathbf{A} - \mathbf{K}_f\mathbf{C}) \begin{bmatrix} \widehat{x} \\ \widehat{b} \end{bmatrix} + [\mathbf{K}_f \quad \mathbf{B}] \begin{bmatrix} x_m \\ v_m \end{bmatrix},$$

$$\text{or:} \quad \boxed{\begin{bmatrix} \dot{\widehat{x}} \\ \dot{\widehat{b}} \end{bmatrix} = \begin{bmatrix} -\sqrt{2q} & -1 \\ q & 0 \end{bmatrix} \begin{bmatrix} \widehat{x} \\ \widehat{b} \end{bmatrix} + \begin{bmatrix} \sqrt{2q} & 1 \\ -q & 0 \end{bmatrix} \begin{bmatrix} x_m \\ v_m \end{bmatrix}}.$$

Question 4. An unbiased estimate of the velocity can be directly obtained by removing the estimate of the bias from the velocity measurement v_m: $\widehat{\dot{x}} = v_m - \widehat{b}$.

Question 5. A state space realization of $F(\mathrm{s})$ is:

$$\begin{cases} \begin{bmatrix} \dot{\widehat{x}} \\ \dot{\widehat{b}} \end{bmatrix} = \begin{bmatrix} -\sqrt{2q} & -1 \\ q & 0 \end{bmatrix} \begin{bmatrix} \widehat{x} \\ \widehat{b} \end{bmatrix} + \begin{bmatrix} \sqrt{2q} & 1 \\ -q & 0 \end{bmatrix} \begin{bmatrix} x_m \\ v_m \end{bmatrix} \\ \begin{bmatrix} \widehat{x} \\ \widehat{\dot{x}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \widehat{x} \\ \widehat{b} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_m \\ v_m \end{bmatrix} \end{cases}$$

The associated transfer matrix $\mathbf{F}(\mathrm{s})$ is:

$$\mathbf{F}(\mathrm{s}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -\sqrt{2q} & -1 \\ q & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} \sqrt{2q} & 1 \\ -q & 0 \end{bmatrix}.$$

That is:

$$\begin{bmatrix} \widehat{X}(\mathrm{s}) \\ \dot{\widehat{X}}(\mathrm{s}) \end{bmatrix} = \frac{\begin{bmatrix} \sqrt{2q}\mathrm{s} + q & \mathrm{s} \\ q\mathrm{s} & \mathrm{s}^2 + \sqrt{2q}\mathrm{s} \end{bmatrix}}{\mathrm{s}^2 + \sqrt{2q}\mathrm{s} + q} \begin{bmatrix} X_m(\mathrm{s}) \\ V_m(\mathrm{s}) \end{bmatrix}. \tag{2.20}$$

Question 6.

$$\frac{\dot{\widehat{X}}}{V_m}(\mathrm{s}) = \frac{\mathrm{s}^2 + \sqrt{2q}\mathrm{s}}{\mathrm{s}^2 + \sqrt{2q}\mathrm{s} + q}.$$

This is a second order high pass filter with a cutting frequency $\sqrt{q}$ (rd/s) and a damping ratio $\sqrt{2}/2$ $(\forall q)$. The steady state (D. C.) gain of this filter is null (the null frequency component, that is the bias, is filtered). The cutting frequency is as greater as q is great, that is, as the bias may derive a lot.
If measures are perfect then $x_m = x$ and $v_m = \dot{x}$. So we have:

- $X_m(\mathrm{s}) = X(\mathrm{s})$ and $V_m(\mathrm{s}) = sX(\mathrm{s})$. Reporting this result in the first row of (2.20), we get:

$$\widehat{X}(\mathrm{s}) = \frac{(\sqrt{2q}s + q)X(\mathrm{s}) + s^2X(\mathrm{s})}{s^2 + \sqrt{2q}s + q} \quad \Rightarrow \quad \widehat{X}(\mathrm{s}) = X(\mathrm{s}) \ .$$

- $V_m(\mathrm{s}) = \dot{X}(\mathrm{s})$ and $X_m(\mathrm{s}) = 1/s\,\dot{X}(\mathrm{s})$. Reporting that in second row of (2.20), we get:

$$\widehat{\dot{X}}(\mathrm{s}) = \frac{\dot{X}(\mathrm{s}) + (s^2 + \sqrt{2q}s)\dot{X}(\mathrm{s})}{s^2 + \sqrt{2q}s + q} \quad \Rightarrow \quad \widehat{\dot{X}}(\mathrm{s}) = \dot{X}(\mathrm{s}) \ .$$

This filter does not induce any degradation of the quality of the measurement. Such a filter is called a **complementary filter**.

□

2.4 Discrete-time Kalman filter

2.4.1 Discrete-time Kalman model

By analogy with the continuous-time case the discrete-time KALMAN model is:

$$\left\{ \begin{array}{l} \mathbf{x}(k+1) = \mathbf{A}_d\mathbf{x}(k) + \mathbf{B}_d\mathbf{u}(k) + M_d\mathbf{w}_d(k) \quad \text{state equation,} \quad \mathbf{x} \in \mathbb{R}^n,\ \mathbf{u} \in \mathbb{R}^m,\ \mathbf{w}_d \in \mathbb{R}^q \\ \mathbf{y}(k) = \mathbf{C}_d\mathbf{x}(k) + \mathbf{D}_d\mathbf{u}(k) + \mathbf{v}_d(k) \quad \text{measurement equation,} \quad \mathbf{y} \in \mathbb{R}^p,\ \mathbf{v}_d \in \mathbb{R}^p \end{array} \right. \tag{2.21}$$

Assumptions : we will assume that:

H1: The pair $(\mathbf{A}_d,\ \mathbf{C}_d)$ is detectable,

H2: Signals $\mathbf{w}_d(k)$ and $\mathbf{v}_d(k)$ are **centered gaussian pseudo-white noises** with **covariance matrices** $\mathbf{W}_d$ and $\mathbf{V}_d$ respectively, that is:

- $\mathrm{E}[\mathbf{w}_d(k)\,\mathbf{w}_d(k+l)^T] = \mathbf{W}_d\,\delta_d(l)$,
- $\mathrm{E}[\mathbf{v}_d(k)\,\mathbf{v}_d(k+l)^T] = \mathbf{V}_d\,\delta_d(l)$
- $\mathrm{E}[\mathbf{w}_d(k)\,\mathbf{v}_d(k+l)^T] = \mathbf{0}$ (with $\delta_d(l) = 1$ if $l = 0$; 0 else).

H3: $\mathbf{V}_d$ is invertible.

Remark 2.1 *: While in the continuous-time case, white noises in the* KALMAN *model are defined by their PSD* $\mathbf{W}$ *and* $\mathbf{V}$ *(variances are infinite), discrete-time* KALMAN *model noises are defined by their covariance matrices* $\mathbf{W}_d$ *and* $\mathbf{V}_d$ *(variances are finite). PSD of these discrete-time noises are constant (and equal to* $dt\,\mathbf{W}_d$ *and* $dt\,\mathbf{V}_d$*, where* dt *is the sampling period) but on a limited range of the frequency* $\omega \in [-\frac{\pi}{dt} \quad \frac{\pi}{dt}]$ *(see appendix B.1); this is why this noise is said to be* ***pseudo****-white.*

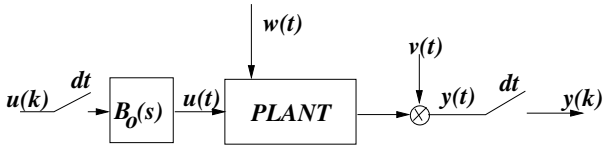

Figure 2.11: Continuous-time plant with zero-order holds on the inputs and sampled outputs.

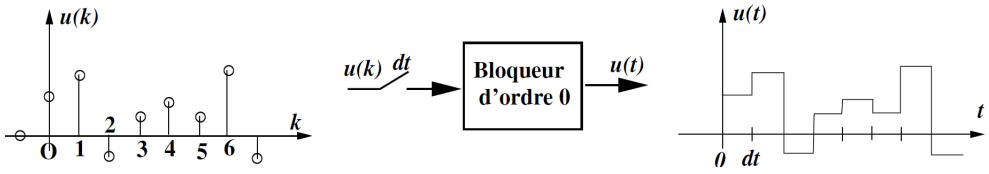

Figure 2.12: The zero-order hold: $B_o(\mathrm{s})$.

Remark 2.2 *: like in the continuous-time case, if the noise* $\mathbf{w}_d$ *is a colored noise and characterized by a spectrum in the* z*-plane* $\mathbf{\Phi_{ww}}(\mathrm{z})$ *the factorization* $\mathbf{\Phi_{ww}}(\mathrm{z}) = \mathbf{G}(\mathrm{z}^{-1})\mathbf{G}^T(\mathrm{z})$ *(see section B.3.2) allows a* Markov *representation of* $\mathbf{w}_d$ *to be derived and to be taken into account in an augmented* Kalman *model.*

2.4.2 A particular case: continuous-time plant with discrete-time measurements

As it was previously mentioned, practical implementation of Kalman filter, even for continuous-time plant, is performed on a numerical computer and in discrete-time. So we will consider that the measurement of the continuous-time plant (1.1) is sampled with a sampling period dt. We will also assume that zero-order holds are placed on deterministic inputs u (see Figures 2.11 and 2.12) and we are going to determine a discrete-time representation of this model.

We denote $\mathbf{x}(k\,dt) = \mathbf{x}(k)$. From the general solution (1.3), the state equation integration between $t_0 = k\,dt$ and $t = (k+1)\,dt$ reads:

$$\mathbf{x}(k{+}1) = \mathrm{e}^{\mathbf{A}dt}\mathbf{x}(k) + \left(\int_{k\,dt}^{(k+1)dt} \mathrm{e}^{\mathbf{A}((k+1)dt-\tau)}\mathbf{B}d\tau\right)\mathbf{u}(k) + \int_{k\,dt}^{(k+1)dt} \mathrm{e}^{\mathbf{A}((k+1)dt-\tau)}\mathbf{M}\mathbf{w}(\tau)d\tau\ .$$

The change of variable: $(k+1)dt - \tau = v$ leads to the following results:

$$\mathbf{x}(k+1) = \mathrm{e}^{\mathbf{A}dt}\mathbf{x}(k) + \left(\int_0^{dt} \mathrm{e}^{\mathbf{A}v}\mathbf{B}dv\right)\mathbf{u}(k) + \int_0^{dt} \mathrm{e}^{\mathbf{A}v}\mathbf{M}\mathbf{w}((k+1)dt - v)dv\ .$$

The measurement equation is a static equation and then, at each sampling time, we

can write:

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) + \mathbf{v}(k)$$

The discrete time state equation is expressed under the form of (2.21) with:

$$\boxed{\mathbf{A}_d = \mathrm{e}^{\mathbf{A}dt}, \quad \mathbf{B}_d = \int_0^{dt} \mathrm{e}^{\mathbf{A}v}\mathbf{B}dv, \quad \mathbf{M}_d = \mathbf{1}_n, \quad \mathbf{C}_d = \mathbf{C}, \quad \mathbf{D}_d = \mathbf{D}} \tag{2.22}$$

The discrete-time state and measurement noises read:

$$\mathbf{w}_d(k) = \int_0^{dt} \mathrm{e}^{\mathbf{A}v}\mathbf{M}\mathbf{w}((k+1)dt - v)dv, \quad \mathbf{v}_d(k) = \mathbf{v}(kdt)\ ,$$

It is required to characterized them by their respective covariance matrices $\mathbf{W}_d$ and $\mathbf{V}_d$.

Due to sampling, the covariance matrix of discrete-time measurement noise becomes:

$$\boxed{\mathbf{V}_d = \mathbf{V}/dt}\ . \tag{2.23}$$

Justification: The continuous-time measurement equation $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{v}(t)$ involves a white noise $\mathbf{v}(t)$ with an infinite variance and a finite PSD $\mathbf{V}$. The sampling of the measurement with a sampling step dt provides a numerical series $\mathbf{y}(k)$ with an infinite variance and makes the KALMAN filter design to be singular. The frequency domain response (or the PSD) of noise $\mathbf{v}(t)$ must be limited between $-\pi/dt$ and π/dt to sample correctly the measurement $\mathbf{y}(t)$. One can recognize an analogy with the SHANNON condition for deterministic signals; that is: the sampling cannot allow to take into account the signal components with a frequency higher than π/dt. The PSD of the continuous-time noise, limited in frequency domain, is depicted in Figure 2.13. The variance of such a random signal is now finite and is equal to (see remark 1.3):

$$\mathbf{var}_{\mathbf{v}} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathbf{\Phi}_{vv}(\omega)d\omega = \frac{\mathbf{V}}{2\pi}\int_{-\frac{\pi}{dt}}^{\frac{\pi}{dt}} d\omega = \frac{\mathbf{V}}{dt}\ .$$

As the sampling of a signal does not change its variance, the discrete-time measurement equation is: $\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) + \mathbf{v}_d(k)$ with $\mathrm{E}[\mathbf{v}_d(k)\,\mathbf{v}_d(k+l)^T] = \frac{\mathbf{V}}{dt}\,\delta_d(l)$.

□

Now we have to compute the covariance matrix $\mathbf{W}_d$ of the state noise $\mathbf{w}_d(k)$:

$$\begin{aligned}
\mathbf{W}_d &= \mathrm{E}[\mathbf{w}_d(k)\mathbf{w}_d^T(k)] = E\left[\int_0^{dt} \mathrm{e}^{\mathbf{A}v}\mathbf{M}\mathbf{w}((k+1)dt - v)dv \int_0^{dt} \mathbf{w}^T((k+1)dt - \tau)\mathbf{M}^T\mathrm{e}^{\mathbf{A}^T\tau}d\tau\right] \\
&= \int\int_0^{dt} \mathrm{e}^{\mathbf{A}v}\mathbf{M}\mathrm{E}[\mathbf{w}((k+1)dt - v)\mathbf{w}^T((k+1)dt - \tau)]\mathbf{M}^T\mathrm{e}^{\mathbf{A}^T\tau}dvd\tau \\
&= \int\int_0^{dt} \mathrm{e}^{\mathbf{A}v}\mathbf{M}\mathbf{W}\delta(\tau - v)\mathbf{M}^T\mathrm{e}^{\mathbf{A}^T\tau}dvd\tau
\end{aligned}$$

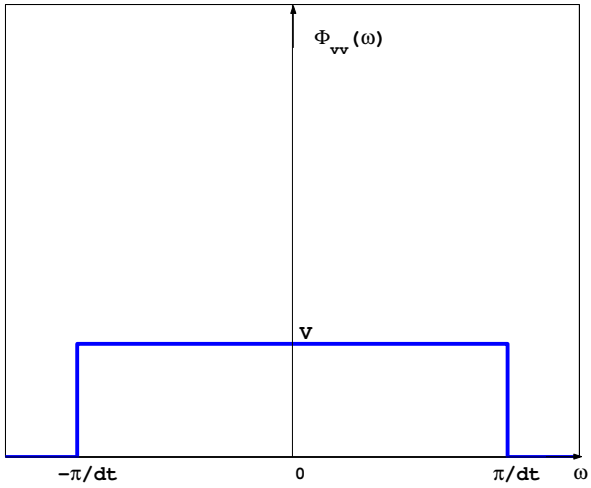

Figure 2.13: Frequency-domain limitation of the continuous-time measurement noise.

$$\boxed{\mathbf{W}_d = \int_0^{dt} \mathrm{e}^{\mathbf{A}v}\mathbf{M}\mathbf{W}\mathbf{M}^T\mathrm{e}^{\mathbf{A}^T v}dv} \,. \tag{2.24}$$

Remark 2.3 *: We mention in appendix B (see remark B.2, equation (B.13)) how* $\mathbf{W}_d$ *can be computed; but we can also use the approximation* $\mathbf{W}_d \approx dt\mathbf{M}\mathbf{W}\mathbf{M}^T$ *if* dt *is small with respect to the settling time of the plant. Lastly, we have to keep in mind that all these formulaes are embedded in Matlab macro-functions* `lqed` *or* `kalmd`*. These functions allow a discrete-time* Kalman *filter to be designed directly from the continuous-time data (*$\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$*,* $\mathbf{D}$*,* $\mathbf{M}$*,* $\mathbf{W}$ *and* $\mathbf{V}$*) and a sampling period* dt *(see the Matlab illustration in appendix D.3). Such an approach (and such functions) cannot support correlation between measurement and state noises.*

2.4.3 Recurrent equations of discrete Kalman filter

The discrete Kalman filter principle is the same than in the continuous-time case. It involves a prediction based on the deterministic model and a correction (updating) based on the innovation (difference between the measurement and its prediction) but in the discrete-time case, we will distinguish:

- the **predicted state** at time $k+1$, denoted $\widehat{\mathbf{x}}(k+1|k)$, knowing all the measures until time k; which is associated with the **prediction error covariance** matrix denoted:

$$\mathbf{P}(k+1|k) = \mathrm{E}\Big[\Big(\mathbf{x}(k+1) - \widehat{\mathbf{x}}(k+1|k)\Big)\Big(\mathbf{x}(k+1) - \widehat{\mathbf{x}}(k+1|k)\Big)^T\Big] \,.$$

- the **estimated (or updated) state**, denoted $\widehat{\mathbf{x}}(k+1|k+1)$, knowing all the measures until time $k+1$ (just after the correction or the updating); which is associated with the **estimation error covariance** matrix, denoted:

$$\mathbf{P}(k+1|k+1) = \mathrm{E}\Big[\Big(\mathbf{x}(k+1) - \widehat{\mathbf{x}}(k+1|k+1)\Big)\Big(\mathbf{x}(k+1) - \widehat{\mathbf{x}}(k+1|k+1)\Big)^T\Big] .$$

Prediction: at time k, we know $\widehat{\mathbf{x}}(k|k)$ and we predict the state at time $k+1$ using the deterministic model

$$\boxed{\widehat{\mathbf{x}}(k+1|k) = \mathbf{A}_d\widehat{\mathbf{x}}(k|k)+\mathbf{B}_d\mathbf{u}(k)} . \tag{2.25}$$

At time k, the estimation error was characterized by $\mathbf{P}(k|k)$. The prediction model being wrong, the error can only increase and this error at time $k+1$ will be characterized by (see theorem B.1 in appendix) :

$$\boxed{\mathbf{P}(k+1|k) = \mathbf{A}_d\mathbf{P}(k|k)A_d^T + \mathbf{M}_d\mathbf{W}_d\mathbf{M}_d^T} \tag{2.26}$$

Correction: at time $k+1$, the prediction is updated with the innovation through the filter gain $\mathbf{K}_f(k+1)$:

$$\boxed{\widehat{\mathbf{x}}(k+1|k+1) = \widehat{\mathbf{x}}(k+1|k) + \mathbf{K}_f(k+1)\Big(\mathbf{y}(k+1) - \mathbf{C}_d\widehat{\mathbf{x}}(k+1|k) - \mathbf{D}_d\mathbf{u}(k+1)\Big)} . \tag{2.27}$$

Using the measurement equation of model (2.21), we can write:

$$\mathbf{x}(k+1)-\widehat{\mathbf{x}}(k+1|k+1) = \Big(\mathbf{1}_n-\mathbf{K}_f(k+1)\mathbf{C}_d\Big)(\mathbf{x}(k+1)-\widehat{\mathbf{x}}(k+1|k))-\mathbf{K}_f(k+1)\mathbf{v}_d(k+1) \quad \text{and :}$$

$$\begin{aligned}
\mathbf{P}(k+1|k+1) &= \Big(\mathbf{1}_n - \mathbf{K}_f(k+1)\mathbf{C}_d\Big)\mathbf{P}(k+1|k)\Big(\mathbf{1}_n - \mathbf{K}_f(k+1)\mathbf{C}_d\Big)^T + \mathbf{K}_f(k+1)\mathbf{V}_d\mathbf{K}_f^T(k+1) \\
&= \mathbf{P}(k+1|k) - \mathbf{K}_f(k+1)\mathbf{C}_d\mathbf{P}(k+1|k) - \mathbf{P}(k+1|k)\mathbf{C}_d^T\mathbf{K}_f^T(k+1)\cdots \\
&\quad \cdots + \mathbf{K}_f(k+1)\Big(\mathbf{C}_d\mathbf{P}(k+1|k)\mathbf{C}_d^T + \mathbf{V}_d\Big)\mathbf{K}_f^T(k+1) .
\end{aligned} \tag{2.28}$$

Like in the continuous-time case, we are looking for $\mathbf{K}_f(k+1)$ which minimizes $\mathbf{P}_{(i,i)}(k+1|k+1)$, $\forall\, i = 1, \ldots, n$:

$$\frac{\partial\mathbf{P}_{(i,i)}(k+1|k+1)}{\partial\mathbf{K}_{f(i,:)}(k+1)} = -2\mathbf{P}_{(i,:)}(k+1|k)\mathbf{C}_d^T + 2\mathbf{K}_{f(i,:)}(k+1)\Big(\mathbf{C}_d\mathbf{P}(k+1|k)\mathbf{C}_d^T + \mathbf{V}_d\Big) .$$

One can deduce:

$$\boxed{\mathbf{K}_f(k+1) = \mathbf{P}(k+1|k)\mathbf{C}_d^T\Big(\mathbf{C}_d\mathbf{P}(k+1|k)\mathbf{C}_d^T + \mathbf{V}_d\Big)^{-1}} . \tag{2.29}$$

Returning this expression in 2.28, we get:

$$\boxed{\mathbf{P}(k+1|k+1) = \Big(\mathbf{1}_n - \mathbf{K}_f(k+1)\mathbf{C}_d\Big)\mathbf{P}(k+1|k)} \,. \tag{2.30}$$

The set of equations (2.25), (2.26), (2.27), (2.29) and (2.30) constitutes the discrete-time Kalman filter. Equations (2.25) and (2.27) are initialized with $\widehat{\mathbf{x}}(0|0)$, the initial estimate and equations (2.26), (2.29) and (2.30) are initialized with $\mathbf{P}(0|0)$, the confidence we have in the initialization.

If the model and noises are stationary, equations (2.26), (2.29) and (2.30) can be integrated off-line. Removing (2.29) and (2.30) in (2.26), one can find a recurrent Riccati equation in the prediction error covariance:

$$\mathbf{P}(k+1|k) = \mathbf{A}_d\mathbf{P}(k|k-1)\mathbf{A}_d^T - \mathbf{A}_d\mathbf{P}(k|k-1)\mathbf{C}_d^T\Big(\mathbf{C}_d\mathbf{P}(k|k-1)\mathbf{C}_d^T + \mathbf{V}_d\Big)^{-1}\mathbf{C}_d\mathbf{P}(k|k-1)\mathbf{A}_d^T + \mathbf{M}_d\mathbf{W}_d\mathbf{M}_d^T \,. \tag{2.31}$$

Lastly, in steady state, $\mathbf{K}_f(k+1) = \mathbf{K}_f(k) = \mathbf{K}_f$, but on can distinguish:

- $\mathbf{P}_p = \mathbf{P}(k+1|k) = \mathbf{P}(k|k-1) = \cdots$: the prediction error covariance matrix in steady state which is the positive solution of the discrete-time algebraic Riccati equation:

$$\boxed{\mathbf{P}_p = \mathbf{A}_d\mathbf{P}_p\mathbf{A}_d^T - \mathbf{A}_d\mathbf{P}_p\mathbf{C}_d^T(\mathbf{C}_d\mathbf{P}_p\mathbf{C}_d^T + \mathbf{V}_d)^{-1}\mathbf{C}_d\mathbf{P}_p\mathbf{A}_d^T + \mathbf{M}_d\mathbf{W}_d\mathbf{M}_d^T} \,. \tag{2.32}$$

- $\mathbf{P}_e = \mathbf{P}(k+1|k+1) = \mathbf{P}(k|k) = \cdots$: the estimation error covariance matrix in steady state:

$$\mathbf{P}_e = (\mathbf{1}_n - \mathbf{K}_f\mathbf{C}_d)\mathbf{P}_p \,.$$

Then, the state space realization of the stationary (steady state) Kalman filter is (with (2.27) in (2.25)):

$$\boxed{\left\{\begin{array}{rcll} \widehat{\mathbf{x}}(k+1|k) & = & \mathbf{A}_d(\mathbf{1}_n - \mathbf{K}_f\mathbf{C}_d)\widehat{\mathbf{x}}(k|k-1) & +[\mathbf{A}_d\mathbf{K}_f \quad \mathbf{B}_d - \mathbf{A}_d\mathbf{K}_f\mathbf{D}_d]\left[\begin{array}{c}\mathbf{y}(k)\\ \mathbf{u}(k)\end{array}\right] \\ \widehat{\mathbf{x}}(k|k) & = & (\mathbf{1}_n - \mathbf{K}_f\mathbf{C}_d)\widehat{\mathbf{x}}(k|k-1) & +[\mathbf{K}_f \quad -\mathbf{K}_f\mathbf{D}_d]\left[\begin{array}{c}\mathbf{y}(k)\\ u(k)\end{array}\right] \end{array}\right.} \tag{2.33}$$

The state of this realization is the predicted state, the output is the estimated state.

Remark 2.4 *:*

- $0 < \mathbf{P}_e < \mathbf{P}_p$. *Indeed, from (2.29) and (2.30), we get:*

$$\mathbf{P}_e = \mathbf{P}_p - \mathbf{P}_p\mathbf{C}_d^T(\mathbf{C}_d\mathbf{P}_p\mathbf{C}_d^T + \mathbf{V}_d)^{-1}\mathbf{C}_d\mathbf{P}_p \ .$$

  The second term of the right-hand member is always positive, thus: $\mathbf{P}_e < \mathbf{P}_p$, *that is the estimation error variance is always lower than the prediction error variance (or the state noise variance propagated in the state equation).*

- *Lastly the (un-noised) output* $\mathbf{y}_p(k) = \mathbf{C}_d\mathbf{x}(k) + \mathbf{D}_d\mathbf{u}(k)$ *can be estimated by* $\widehat{\mathbf{y}}_p(k) = \mathbf{C}_d\widehat{\mathbf{x}}(k|k) + \mathbf{D}_d\mathbf{u}(k)$. *The estimation error covariance for this output reads:*

$$\mathbf{C}_d\mathbf{P}_e\mathbf{C}_d^T = \mathbf{C}_d\mathbf{P}_p\mathbf{C}_d^T - \mathbf{C}_d\mathbf{P}_p\mathbf{C}_d^T(\mathbf{C}_d\mathbf{P}_p\mathbf{C}_d^T + \mathbf{V}_d)^{-1}\mathbf{C}_d\mathbf{P}_p\mathbf{C}_d^T = \mathbf{C}_d\mathbf{P}_p\mathbf{C}_d^T(\mathbf{C}_d\mathbf{P}_p\mathbf{C}_d^T + \mathbf{V}_d)^{-1}\mathbf{V}_d \ .$$

  That is:

$$\mathbf{C}_d\mathbf{P}_e\mathbf{C}_d^T - \mathbf{V}_d = \Big(\mathbf{C}_d\mathbf{P}_p\mathbf{C}_d^T(\mathbf{C}_d\mathbf{P}_p\mathbf{C}_d^T + \mathbf{V}_d)^{-1} - \mathbf{1}_p\Big)\mathbf{V}_d = -\mathbf{V}_d(\mathbf{C}_d\mathbf{P}_p\mathbf{C}_d^T + \mathbf{V}_d)^{-1}\mathbf{V}_d < 0 \ .$$

  So: $\mathbf{C}_d\mathbf{P}_e\mathbf{C}_d^T < \mathbf{V}_d$ *i.e.* $\widehat{\mathbf{y}}_p(k)$ *is a better estimate of* $\mathbf{y}_p$ *that the direct measurement.*

- *The way to solve Discrete-time Algebraic Riccati Equations (DARE) are not detailed here. Macro-functions (*`lqe` *in Scilab or* `dlqe`, `dare`, `kalman` *in Matlab) allow such equations to be solved, that is: to compute* $\mathbf{K}_f$ *and to provide the state space realization (2.33) of the filter in steady state. The way to use these function is illustrate in the Matlab session presented in appendix D.3.*

Exercise 2.1 *Demonstrate that in the case of a continuous-time plant with discrete-time sampled output (with a sampling period dt),*

- *the continuous-time* KALMAN *filter designed from continuous-time data (*$\mathbf{A}$, $\mathbf{B}$, $\mathbf{M}$, $\mathbf{C}$, $\mathbf{W}$, $\mathbf{V}$*) and then discretized by the* EULER *method*
- *and discrete-time* KALMAN *filter designed from equations (2.22), (2.23) and (2.24)*

tend towards the same solution as dt tends towards 0 *(first order calculus in dt).*

Solution : *The continuous-time* KALMAN *filter is defined by equations (2.8), (2.13) and (2.14). The* EULER *method consists in removing* $\dot{\mathbf{x}}(t)$ *and* $\mathbf{x}(t)$ *by* $\frac{\mathbf{x}(k+1)-\mathbf{x}(k)}{dt}$ *and* $\mathbf{x}(k)$, *respectively. Removing (2.13) in (2.8), this first discrete-time filter is defined by:*

$$\left\{\begin{array}{rcl} \mathbf{P}(k+1) & = & \mathbf{P}(k) + dt\Big(\mathbf{A}\mathbf{P}(k) + \mathbf{P}(k)\mathbf{A}^T - \mathbf{P}(k)\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}(k)\mathbf{P} + \mathbf{M}\mathbf{W}\mathbf{M}^T\Big) \\ \widehat{\mathbf{x}}(k+1) & = & \widehat{\mathbf{x}}(k) + dt\Big(\mathbf{A}\widehat{\mathbf{x}}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{P}(k)\mathbf{C}^T\mathbf{V}^{-1}\Big(\mathbf{y}(k) - \mathbf{C}\widehat{\mathbf{x}}(k) - \mathbf{D}\mathbf{u}(k)\Big)\Big) \end{array}\right. \tag{2.34}$$

The prediction error covariance of the discrete-time Kalman *filter is given by equation (2.31), let us denote* $\mathbf{P}(k|k-1) = \mathbf{P}(k)$*, then using equations (2.22), (2.23) and (2.24) and with the first order approximation* $\mathbf{W}_d \approx dt\mathbf{M}\mathbf{W}\mathbf{M}^T$*, we get:*

$$\mathbf{P}(k+1) \approx \mathrm{e}^{\mathbf{A}dt}\mathbf{P}(k)\mathrm{e}^{\mathbf{A}^T dt} - \mathrm{e}^{\mathbf{A}dt}\mathbf{P}(k)\mathbf{C}^T\Big(\mathbf{C}\mathbf{P}(k)\mathbf{C}^T + \mathbf{V}/dt\Big)^{-1}\mathbf{C}\mathbf{P}(k)\mathrm{e}^{\mathbf{A}^T dt} + dt\mathbf{M}\mathbf{W}\mathbf{M}^T \ .$$

Or with a first order expansion:

$$\begin{aligned}
\mathbf{P}(k+1) &\approx (\mathbf{1}_n + \mathbf{A}dt)\mathbf{P}(k)(\mathbf{1}_n + \mathbf{A}^T dt) + dt\mathbf{M}\mathbf{W}\mathbf{M}^T \\
&\quad - dt(\mathbf{1}_n + \mathbf{A}dt)\mathbf{P}(k)\mathbf{C}^T\Big(\mathbf{1}_p - dt\mathbf{V}^{-1}\mathbf{C}\mathbf{P}(k)\mathbf{C}^T\Big)\mathbf{V}^{-1}\mathbf{C}\mathbf{P}(k)(\mathbf{1}_n + \mathbf{A}^T dt) \\
&\approx \mathbf{P}(k) + dt\Big(\mathbf{A}\mathbf{P}(k) + \mathbf{P}(k)\mathbf{A}^T - \mathbf{P}(k)\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}\mathbf{P}(k) + \mathbf{M}\mathbf{W}\mathbf{M}^T\Big) + dt^2(\cdots) \ .
\end{aligned}$$

So one can recognize (at the first order) the first equation of (2.34). The gain $\mathbf{K}_f$ *becomes:*

$$\mathbf{K}_f(k) = \mathbf{P}(k)\mathbf{C}^T\Big(\mathbf{C}\mathbf{P}(k)\mathbf{C}^T\mathbf{V}^{-1}dt + \mathbf{1}_p\Big)^{-1}\mathbf{V}^{-1}dt \approx \mathbf{P}(k)\mathbf{C}^T\mathbf{V}^{-1}dt,$$

and the state equation of the filter (equation (2.33)) becomes (we denote $\widehat{\mathbf{x}}(k|k-1) = \widehat{\mathbf{x}}(k)$ *and we use also the first order approximation:* $\mathbf{B}_d \approx dt\mathbf{B}$*):*

$$\begin{aligned}
\widehat{\mathbf{x}}(k+1) &= \mathrm{e}^{\mathbf{A}dt}\Big(\mathbf{1}_n - dt\mathbf{P}(k)\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}\Big)\widehat{\mathbf{x}}(k) + dt\mathrm{e}^{\mathbf{A}dt}\mathbf{P}(k)\mathbf{C}^T\mathbf{V}^{-1}\mathbf{y}(k) \\
&\quad + \Big(\mathbf{B}_d - dt\,\mathrm{e}^{\mathbf{A}dt}\mathbf{P}(k)\mathbf{C}^T\mathbf{V}^{-1}\mathbf{D}\Big)\mathbf{u}(k) \\
&\approx (\mathbf{1}_n - \mathbf{A}dt)\Big(\mathbf{1}_n - dt\mathbf{P}(k)\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}\Big)\widehat{\mathbf{x}}(k) + dt(\mathbf{1}_n - \mathbf{A}dt)\mathbf{P}(k)\mathbf{C}^T\mathbf{V}^{-1}\mathbf{y}(k) \\
&\quad + \Big(\mathbf{B}_d - dt(\mathbf{1}_n - \mathbf{A}dt)\mathbf{P}(k)\mathbf{C}^T\mathbf{V}^{-1}\mathbf{D}\Big)\mathbf{u}(k) \\
&\approx \widehat{\mathbf{x}}(k) + dt\Big(\mathbf{A}\widehat{\mathbf{x}}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{P}(k)\mathbf{C}^T\mathbf{V}^{-1}\Big(\mathbf{y}(k) - \mathbf{C}\widehat{\mathbf{x}}(k) - \mathbf{D}\mathbf{u}(k)\Big)\Big) + dt^2(\cdots) \ .
\end{aligned}$$

In the first order approximation, we recognize the second equation (2.34). So both discrete-time filters are equivalent when dt *tends toward* 0*.*

2.4.4 Example

Let us consider again exercise 2.3.2 on the bias estimation. We wish to implement the filter on a numerical computer, both measurements v_m and x_m being sampled with the sampling period dt.

- Provide state-space equations of the discrete-time Kalman model with $[x(k), \quad b(k)]^T$ as state vector.

- Compute using Matlab the gain $\mathbf{K}_f$ and the estimation error covariance in steady state.
  Numerical application: $dt = 0.01\,s$, $V = 1$, $q = 1$.

Correction: The computation of discrete-time model consists in applying formulaes (2.22), (2.23) and (2.24) to the continuous-time model defined by equation (2.19). We assume the velocity measurement $v_m(k)$ constant on a sampling period (that corresponds to introduce a zero-order hold). So we can write:

$$\mathbf{A}_d = \mathrm{e}^{\begin{bmatrix} 0 & -dt \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -dt \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \cdots = \begin{bmatrix} 1 & -dt \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{B}_d = \int_0^{dt} \begin{bmatrix} 1 & -v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} dv = \begin{bmatrix} dt \\ 0 \end{bmatrix},$$

$$\mathbf{C}_d = [1 \quad 0], \quad D_d = 0, \quad \mathbf{V}_d = \frac{1}{dt},$$

$$\mathbf{W}_d = \int_0^{dt} \begin{bmatrix} 1 & -v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & q^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -v & 1 \end{bmatrix} dv = \int_0^{dt} \begin{bmatrix} q^2v^2 & -q^2v \\ -q^2v & q^2 \end{bmatrix} dv$$

$$\Rightarrow \mathbf{W}_d = \begin{bmatrix} \frac{1}{3}q^2\,dt^3 & -\frac{1}{2}q^2\,dt^2 \\ -\frac{1}{2}q^2\,dt^2 & q^2\,dt \end{bmatrix} \left(\approx \begin{bmatrix} 0 & 0 \\ 0 & q^2\,dt \end{bmatrix} \right).$$

The discrete-time KALMAN model is thus defined by:

$$\boxed{\begin{cases} \begin{bmatrix} x(k+1) \\ b(k+1) \end{bmatrix} = \begin{bmatrix} 1 & -dt \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ b(k) \end{bmatrix} + \begin{bmatrix} dt \\ 0 \end{bmatrix} v_m(k) + w_d(k) \\ x_m(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ b(k) \end{bmatrix} + v_d(k) \end{cases}} \tag{2.35}$$

The reader will find in appendix D.3 the Matlab file `demoKalmand` allowing the various model parameters to be computed and also the gain $\mathbf{K}_f$ and the steady state estimation error covariance (from function `dlqe`). We also show how the function `lqed` can be used directly from the data of the continuous-time model.

□

2.5 Exercises

2.5.1 Second order continuous-time system:

A mobile with a mass m is moving along Ox axis due to a force (command) $u(t)$. A perturbation force $w(t)$ acts also on this mass. $w(t)$ is modelled as a gaussian

centered white noise with a PSD W. The position $x(t)$ of this mass is measured. We denote $x_m(t)$ this measurement which is perturbed by a gaussian centered white noise $v(t)$ with a PSD V.

A.N.: $m = 1\,(Kg)$; $\quad W = 1\,(N^2/Hz)$; $\quad V = \rho^2\,(m^2/Hz)$.

From the measurement x_m and the command u, we wish to design a KALMAN filter allowing the position $x(t)$ and the velocity $\dot{x}(t)$ of the mobile to be estimated. These estimates are denoted $\widehat{x}(t)$ and $\widehat{\dot{x}}(t)$.

1) Give the state space equations of the KALMAN model.

2) Compute the KALMAN gain in steady state (as a function of ρ).

3) Compute the transfer matrix $\mathbf{F}(\mathrm{s})$ of the filter:

$$\begin{bmatrix} \widehat{X}(\mathrm{s}) \\ \widehat{\dot{X}}(\mathrm{s}) \end{bmatrix} = \mathbf{F}(\mathrm{s}) \begin{bmatrix} X_m(\mathrm{s}) \\ U(\mathrm{s}) \end{bmatrix}$$

4) Comment this transfer (frequency-domain responses of the various components, influence of ρ).

5) This filter will be implemented in a numerical computer with a sampling period dt. Give the state space equations of the discrete-time model and the recurrent equations of the discrete-time KALMAN filter. We denote $\widehat{x}_0$ and $\widehat{\dot{x}}_0$ the initial position and velocity, and P_0 the initial estimation error covariance.

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Chapter 3

About physical units

It is important to provide physical units for the characteristics of random signals we are going to use or simulate. The aim of this brief chapter is to clarify this point.

First of all, we recall that: if u stands for the physical unit of a random variable $\mathcal{X}$, then its distribution function has no unit and the probability density function $f(x)$ is in u^{-1}.

If u stands for the physical unit of a continuous-time random signal $w(t)$ than physical units of various stochastic characteristics are given in Table 3.1 (equations, allowing the dimension homogeneity to be checked, are also referenced).

Variable	Notation	Physical unit	Equations
signal	$w(t)$	u	
mean	$\mathrm{E}[w(t)]$	u	(1.8) and (1.5)
auto-correlation (variance)	$\phi_{\mathbf{ww}}(t,\tau)$	u^2	(1.9) and (1.6)
spectrum in the s-plane (and PSD)	$\Phi_{\mathbf{ww}}(\mathrm{s})$	$u^2\,s$ or (u^2/Hz)	(1.10)

Table 3.1: Physical units for continuous-time random signal characteristics.

In the case of a discrete-time random signal, the reader will consult Table 3.2 (see also appendix B.1).

One can remark a factor time (s) between the continuous-time PSD (or spectrum in the s-plane) and the discrete-time covariance (or spectrum in the z-plane). Equation (2.23), used for the discretization of the measurement equation of a continuous-time system is homogeneous with that. To characterize a continuous-time noise on a signal expressed in u, some authors specify the square root of the PSD which is thus expressed in $u/\sqrt{Hz}$ (and which is wrongly considered as a standard deviation).

If we consider now the continuous-time KALMAN model (2.1) and if u stands for the physical unit of the state variable $x(t)$, then the physical unit of the state noise $w_x(t) = Mw(t)$ is u/s and its DSP $W_x = MWM^T$ is expressed in u^2/s.

Variable	Notation	Physical unit
signal	$w(n)$	u
mean	$\mathrm{E}[w(n)]$	u
auto-correlation (variance)	$\phi_{\mathbf{ww}}(n,k)$	u^2
spectrum in the z-plane	$\Phi_{\mathbf{ww}}(\mathrm{z})$	u^2
PSD	$\Phi_{\mathbf{ww}}(\omega)$	$u^2 s$ or (u^2/Hz)

Table 3.2: Physical units for discrete-time random signal characteristics.

In the case of discrete-time Kalman filter (2.21), if u stands for the physical unit of $x(k)$ (and also of $x(k+1)$) then $M_d w_d(k)$ is also expressed in u and the covariance of the state noise $M_d W_d M_d^T$ is expressed in u^2. Equation (2.24) used to discretize a continuous-time state noise or the approximation $W_d = W\,dt$ respect also this homogeneity.

References

[1] D. Alazard, C. Cumer, P. Apkarian, M. Gauvrit, and G. Ferreres.
Robustesse et Commande Optimale.
CEPADUES Édition, 1999.

[2] B.D.O. Anderson and J.B. Moore.
Linear optimal control.
Prentice-Hall, 1971.

[3] E.R. Dougherty.
Random processes for image and signal processing.
SPIE Optical Engineering Press, 1999.

[4] H. Kwakernaak and R. Sivan.
Linear optimal control system.
John Wiley and Sons, 1972.

[5] M. Labarrère, J. P. Krief, and B. Gimonet.
Le filtrage et ses applications.
CEPADUES Édition, 1982.

[6] J.L. Pac.
Les processus stochastiques.
polycopié SUPAERO.

[7] Many thanks to.
colleagues form ONERA, ISAE-SUPAERO and Johan Montel (CNES) for constructive discussions and advises.

[8] Wikipedia:.
http://en.wikipedia.org/wiki/Matrix_exponential.

[9] Wikipedia:.
http://en.wikipedia.org/wiki/Riccati_differential_equation.

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Appendix A

State space equation integration

A.1 Continuous-time case

Let us consider a continuous state space model:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \tag{A.1}$$

The response of this model to deterministic input on the time range $t \in [t_0,\ t]$ and to initial conditions $\mathbf{x}(t_0)$ is:

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} \mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\, d\tau \tag{A.2}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{A.3}$$

Proof: the general solution of the state equation "without second member" $\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{0}$ is:

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{K}\ .$$

A particular solution can be derived using the "varying constant method" ($\mathbf{K} \rightarrow \mathbf{K}(t)$) :

$$\mathbf{A}\mathrm{e}^{\mathbf{A}t}\mathbf{K}(t) + \mathrm{e}^{\mathbf{A}t}\dot{\mathbf{K}}(t) = \mathbf{A}\mathrm{e}^{\mathbf{A}t}\mathbf{K}(t) + \mathbf{B}\mathbf{u}(t) \tag{A.4}$$

$$\dot{\mathbf{K}}(t) = \mathrm{e}^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \tag{A.5}$$

$$\mathbf{K}(t) = \int_{t_0}^{t} \mathrm{e}^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\, d\tau + \mathbf{K}_0 \tag{A.6}$$

$$\Rightarrow \quad \mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{K}_0 + \int_{t_0}^{t} \mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\, d\tau\ . \tag{A.7}$$

Taking into account the initial condition at $t = t_0$ yields to $\mathbf{K}_0 = \mathrm{e}^{-\mathbf{A}t_0}\mathbf{x}(t_0)$ and allows to find a unique solution (A.2). Lastly, the output equation is static : $\mathbf{y}(t)$=$\mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$.

□

Such a solution involves the exponential matrix $e^{\mathbf{A}}(t-t_0)$ ([8]) which can be computed using a eigen-value/eigen-vector decomposition of matrix $\mathbf{A}$ if $\mathbf{A}$ is diagonalizable (see example in section C.2 or using a power series expansion (see example below).

Remark A.1 *Using (A.2) with t_0=0, $\mathbf{x}_0 = \mathbf{0}$ and $\mathbf{u}(t) = \boldsymbol{\delta}(t)$ (DIRAC function), the impulse response of the system defined by ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$) is:*

$$\mathbf{f}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}\boldsymbol{\delta}(t) \quad \forall t \geq 0 \quad (\mathbf{f}(t) = \mathbf{0} \text{ si } t < 0) \ .$$

The LAPLACE transform: $\mathbf{F}(\mathrm{s}) = \int_0^{+\infty} \mathbf{f}(\tau)e^{-\tau s}d\tau$ allows to find the well-known result:

$$\mathbf{F}(\mathrm{s}) = \int_0^{+\infty} (\mathbf{C}e^{\mathbf{A}\tau}\mathbf{B}+\mathbf{D}\boldsymbol{\delta}(\tau))e^{-\tau s}d\tau = \mathbf{C}(\mathrm{s}\mathbf{1}_n-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D} \quad (\forall s \in \text{convergence domain}). \tag{A.8}$$

Example A.1 *Let us consider the second order model: $\frac{Y(\mathrm{s})}{U(\mathrm{s})} = \frac{1}{(s+1)^2}$.*

- *a) compute the system impulse response,*
- *b) compute the response of the system to initial conditions $y(0) = y_0$ and $\dot{y}(0) = \dot{y}_0$ (with $t_0 = 0$).*

***Correction**: If the use of LAPLACE transform pair table allows the question a) to be solved directly: $y(t) = te^{-t}$, it is strongly recommended to use a state space approach and equation (A.2) to solve question b). Let us consider a JORDAN realization of the plant:*

$$\begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} \end{cases} . \tag{A.9}$$

As the dynamic matrix $\mathbf{A}$ is not diagonalizable (one multiple eigenvalue of order 2), the computation of the matrix exponential $e^{\mathbf{A}t}$ can be performed using a power series expansion:

$$e^{\mathbf{A}t} = \mathbf{1}_n + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \frac{\mathbf{A}^4t^4}{4!} + \cdots \ .$$

Then we get:

$$e^{\begin{bmatrix} -t & t \\ 0 & -t \end{bmatrix}} = \begin{bmatrix} 1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\cdots & t-t^2+\frac{t^3}{2!}-\frac{t^4}{3!}+\cdots \\ 0 & 1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\cdots \end{bmatrix} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix} .$$

Impulse response: with $t_0 = 0$, $x(t_0) = 0$ *and* $u(t) = \delta(t)$ *(*DIRAC *function), equation (A.2) becomes :*

$$\mathbf{x}(t) = \int_0^t \begin{bmatrix} (t-\tau)\mathrm{e}^{\tau-t} \\ \mathrm{e}^{\tau-t} \end{bmatrix} \delta(\tau)d\tau = \begin{bmatrix} t\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} \quad et \quad \mathbf{y}(t) = [1 \quad 0]x(t) = t\mathrm{e}^{-t} \ .$$

Response to initial conditions $(u(t) = 0)$*: a relationship between the state vector* $x = [x_1,\ x_2]^T$ *and the vector composed of the output* y *and its derivative* $\dot{y}$*, on which the initial conditions are formulated, must be established. The measurement equation and its derivation allow to write:*

$$\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \mathbf{x} \quad \Rightarrow \quad \mathbf{x}(t_0) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ \dot{y}_0 \end{bmatrix} \ .$$

So:

$$y(t) = \mathbf{C}\mathrm{e}^{\mathbf{A}t}x(t_0) = [1 \quad 0] \begin{bmatrix} \mathrm{e}^{-t} & t\mathrm{e}^{-t} \\ 0 & \mathrm{e}^{-t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ \dot{y}_0 \end{bmatrix} = \mathrm{e}^{-t}(1+t)y_0 + t\mathrm{e}^{-t}\dot{y}_0 \ .$$

Remark: an horizontal companion realization can be also used: then the state vector is composed of the output and its derivative:

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \ . \tag{A.10}$$

The computation of the matrix exponential is now:

$$\mathrm{e}^{\begin{bmatrix} 0 & t \\ -t & -2t \end{bmatrix}} = \begin{bmatrix} \mathrm{e}^{-t}(1+t) & t\mathrm{e}^{-t} \\ -t\mathrm{e}^{-t} & \mathrm{e}^{-t}(1-t) \end{bmatrix} \ .$$

□

A.2 Discrete-time case

Let us consider a discrete-time state space model:

$$\begin{cases} \mathbf{x}(k+1) &= \mathbf{A}_d\mathbf{x}(k) + \mathbf{B}_d\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}_d\mathbf{x}(k) + \mathbf{D}_d\mathbf{u}(k) \end{cases} \tag{A.11}$$

The response to deterministic inputs $\mathbf{u}(.)$ over a time range $[k_0,\ k]$ and to initial conditions $\mathbf{x}(k_0) = x_0$ is:

$$\mathbf{x}(k) = \mathbf{A}_d^{k-k_0}\mathbf{x}_0 + \sum_{i=k_0}^{k-1} \mathbf{A}_d^{i-k_0}\mathbf{B}_d\mathbf{u}(k-1-i+k_0) \tag{A.12}$$

$$\mathbf{y}(k) = \mathbf{C}_d\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) \ . \tag{A.13}$$

The proof is directly given solving the state-space recurrent equation:

$$\begin{aligned}\mathbf{x}(k) &= \mathbf{A}_d\mathbf{x}(k-1)+\mathbf{B}_d\mathbf{u}(k-1)\\ &= \mathbf{A}_d^2\mathbf{x}(k-2)+\mathbf{A}_d\mathbf{B}_d\mathbf{u}(k-2)+\mathbf{B}_d\mathbf{u}(k-1)\\ &= \cdots\end{aligned}$$

Remark A.2 *Using (A.12) with k_0=0, $\mathbf{x}_0 = 0$ and $\mathbf{u}(i) = \boldsymbol{\delta}_d(i)$ (discrete Dirac function: $\boldsymbol{\delta}_d(0) = \mathbf{1}_m$; $\boldsymbol{\delta}_d(i) = \mathbf{0}$ if $i \neq 0$), the impulse response of the system defined by ($\mathbf{A}_d$, $\mathbf{B}_d$, $\mathbf{C}_d$, $\mathbf{D}_d$) is therefore:*

$$\mathbf{f}(0) = \mathbf{D}_d, \quad \mathbf{f}(i) = \mathbf{C}_d\mathbf{A}_d^{i-1}\mathbf{B}_d \quad \forall i \geq 1 \quad (\mathbf{f}(i) = \mathbf{0} \text{ if } i < 0)\ .$$

The $\mathcal{Z}$-transform: $\mathbf{F}(\mathrm{z}) = \sum_{i=0}^{\infty}\mathbf{f}(i)\mathrm{z}^{-i}$ allows to find the well-known result:

$$\mathbf{F}(\mathrm{z}) = \sum_{i=1}^{\infty}\mathbf{C}_d\mathbf{A}_d^{i-1}\mathbf{B}_d\mathrm{z}^{-i}+\mathbf{D}_d = \mathbf{C}_d(\mathrm{z}\mathbf{1}_n-\mathbf{A}_d)^{-1}\mathbf{B}_d+\mathbf{D}_d \quad (\forall \mathrm{z} \in \text{domaine de convergence}). \tag{A.14}$$

Appendix B

Transmission of random signals and noises in linear systems

The following proofs are extracted from reference [5] and adapted to the notation of this document.

B.1 Additional background: discrete random signals

The notions introduced in chapter 1 are extended here to the discrete-time case .

Let $\mathbf{w}(n)$ be a sequence of random variables.

- Expected value (mean) : $\mathbf{m}(n) = \mathrm{E}[\mathbf{w}(n)]$.
- Autocorrelation function : $\boldsymbol{\phi}_{\mathbf{ww}}(n,k) = \mathrm{E}[\mathbf{w}(n)\mathbf{w}^T(n+k)]$.
- Wide-sense stationarity: $\mathbf{m}(n) = \mathbf{m}$; $\quad \boldsymbol{\phi}_{\mathbf{ww}}(n,k) = \boldsymbol{\phi}_{\mathbf{ww}}(k) \quad \forall n$.
- Variance : $\mathbf{var_w} = \phi_{\mathbf{ww}}(k)|_{k=0}$.
- Spectrum in the z-plane :

$$\boldsymbol{\Phi}_{\mathbf{ww}}(\mathrm{z}) = \sum_{k=-\infty}^{\infty} \boldsymbol{\phi}_{\mathbf{ww}}(k)\mathrm{z}^{-k} \text{ , and reciprocally :}$$

$$\boldsymbol{\phi}_{\mathbf{ww}}(k) = \frac{1}{2\pi j}\int_{\text{unit circle}} \boldsymbol{\Phi}_{\mathbf{ww}}(\mathrm{z})\mathrm{z}^{k-1}d\mathrm{z} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \boldsymbol{\Phi}_{\mathbf{ww}}(\mathrm{e}^{j\theta})\mathrm{e}^{j\theta k}d\theta \ .$$

- Power Spectral Density (PSD): this function of the frequency ω assumes the introduction of a time scale. So, the sampling period dt between each sample of the sequence $\mathbf{w}(n)$ is introduced.

$$\boldsymbol{\Phi}_{\mathbf{ww}}(\omega) = dt\, \boldsymbol{\Phi}_{\mathbf{ww}}(\mathrm{z})|_{\mathrm{z}=\mathrm{e}^{j\omega dt}}$$

Then, we can state:

$$\boldsymbol{\phi}_{\mathbf{ww}}(k) = \frac{1}{2\pi}\int_{-\pi/dt}^{\pi/dt} \boldsymbol{\Phi}_{\mathbf{ww}}(\omega) \mathrm{e}^{j\omega\, dt\, k} d\omega \quad \text{and} \quad \mathbf{var}_{\mathbf{w}} = \frac{1}{2\pi}\int_{-\pi/dt}^{\pi/dt} \boldsymbol{\Phi}_{\mathbf{ww}}(\omega) d\omega \ .$$

The variance is equal (with a factor $1/2/\pi$) to the integral of the PSD between $-\pi/dt$ et π/dt.

B.2 Time-domain approach

B.2.1 Continuous-time case

Theorem 1.1 (recall of page 21). Let us consider the continuous-time linear system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{M}\mathbf{w}(t) \ . \tag{B.1}$$

$\mathbf{w}(t)$ is a centered gaussian white noise with a PSD $\mathbf{W}$. Let us denote $\mathbf{m}(t_0)$ and $\mathbf{P}(t_0)$ the mean vector and the covariance matrix of the initial state $\mathbf{x}(t_0)$ (also a gaussian random variable independent of $\mathbf{w}(t)$). It is shown that $\mathbf{x}(t)$ is a gaussian random signal:

- with mean vector:
$$\mathbf{m}(t) = \mathrm{E}[\mathbf{x}(t)] = \mathrm{e}^{\mathbf{A}(t-t_0)}\mathbf{m}(t_0)$$
- and a covariance matrix $\mathbf{P}(t) = \mathrm{E}[(\mathbf{x}(t) - \mathbf{m}(t))(\mathbf{x}(t) - \mathbf{m}(t))^T]$ solution of the differential Lyapunov equation:
$$\dot{\mathbf{P}}(t) = \mathbf{A}\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}^T + \mathbf{M}\mathbf{W}\mathbf{M}^T \ . \tag{B.2}$$
If the system is stable (all the eigenvalues of A have a negative real part) a steady state is reached: $\dot{\mathbf{P}} = 0$ and $\mathbf{P}(t) = \mathbf{P}$ is then solution of the continuous-time Lyapunov equation:
$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T + \mathbf{M}\mathbf{W}\mathbf{M}^T = \mathbf{0} \ . \tag{B.3}$$

Proof: The integration of equation (B.1) from initial time t_0 and running time t yields to:

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} \mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{M}\mathbf{w}(\tau) d\tau$$

$\mathbf{x}(t)$ is thus a combination of random gaussian signals ($\mathbf{x}(t_0)$ and $\mathbf{w}(\tau)$), so $\mathbf{x}(t)$ is also a gaussian random signal. Let us compute its mean $\mathbf{m}(t) = \mathrm{E}(\mathbf{x}(t))$ and its covariance matrix $\mathbf{P}(t) = \mathrm{E}[(\mathbf{x}(t) - \mathbf{m}(t))(\mathbf{x}(t) - \mathbf{m}(t))^T]$.

Mean $\mathbf{m}(t)$:

$$\mathbf{m}(t) = \mathrm{e}^{\mathbf{A}(t-t_0)}\mathrm{E}[\mathbf{x}(t_0)] + \int_{t_0}^{t} \mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{M}\,\mathrm{E}[\mathbf{w}(\tau)]d\tau \ ,$$

$\mathrm{E}[\mathbf{w}(\tau)] = \mathbf{0}$ (centered noise) and $\mathrm{E}[\mathbf{x}(t_0)] = \mathbf{m}(t_0)$ so:

$$\boxed{\mathbf{m}(t) = \mathrm{e}^{\mathbf{A}(t-t_0)}\mathbf{m}(t_0)} \ .$$

Covariance $\mathbf{P}(t)$:

$$\begin{aligned}\mathbf{x}(t) - \mathbf{m}(t) &= \mathrm{e}^{\mathbf{A}(t-t_0)}\Big(\mathbf{x}(t_0) - \mathbf{m}(t_0)\Big) + \int_{t_0}^{t} \mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{M}\mathbf{w}(\tau)d\tau \\ &= \mathrm{e}^{\mathbf{A}(t-t_0)}\Big(\mathbf{x}(t_0) - \mathbf{m}(t_0) + \int_{t_0}^{t} \mathrm{e}^{\mathbf{A}(t_0-\tau)}\mathbf{M}\mathbf{w}(\tau)d\tau\Big) \ . \qquad \text{(B.4)}\end{aligned}$$

$$\begin{aligned}\Big(\mathbf{x}(t) - \mathbf{m}(t)\Big)\Big(\mathbf{x}(t) - \mathbf{m}(t)\Big)^T = \mathrm{e}^{\mathbf{A}(t-t_0)}\Bigg(&\Big(\mathbf{x}(t_0) - \mathbf{m}(t_0)\Big)\Big(\mathbf{x}(t_0) - \mathbf{m}(t_0)\Big)^T + \\ &+ \int_{t_0}^{t} \mathrm{e}^{\mathbf{A}(t_0-\tau)}\mathbf{M}\mathbf{w}(\tau)\Big(\mathbf{x}(t_0) - \mathbf{m}(t_0)\Big)^T d\tau + \\ &+ \int_{t_0}^{t} \Big(\mathbf{x}(t_0) - \mathbf{m}(t_0)\Big)\mathbf{w}^T(\tau)\mathbf{M}^T\mathrm{e}^{\mathbf{A}^T(t_0-\tau)}d\tau + \\ &+ \int\int_{t_0}^{t} \mathrm{e}^{\mathbf{A}(t_0-\tau)}\mathbf{M}\mathbf{w}(\tau)\mathbf{w}^T(u)\mathbf{M}^T\mathrm{e}^{\mathbf{A}^T(t_0-u)}d\tau du\Bigg)\mathrm{e}^{\mathbf{A}^T(t-t_0)} \ . \qquad \text{(B.5)}\end{aligned}$$

$$\begin{aligned}\mathbf{P}(t) \;=\; \mathrm{e}^{\mathbf{A}(t-t_0)}\Bigg(&\mathbf{P}(t_0) + \int_{t_0}^{t} \mathrm{e}^{\mathbf{A}(t_0-\tau)}\mathbf{M}\,\mathrm{E}\Big[\mathbf{w}(\tau)\Big(\mathbf{x}(t_0) - \mathbf{m}(t_0)\Big)^T\Big]d\tau + \\ &\int_{t_0}^{t} \mathrm{E}\Big[\Big(\mathbf{x}(t_0) - \mathbf{m}(t_0)\Big)\mathbf{w}^T(\tau)\Big]\mathbf{M}^T\mathrm{e}^{\mathbf{A}^T(t_0-\tau)}d\tau + \\ &\int\int_{t_0}^{t} \mathrm{e}^{\mathbf{A}(t_0-\tau)}\mathbf{M}\,\mathrm{E}\Big[\mathbf{w}(\tau)\mathbf{w}^T(u)\Big]\mathbf{M}^T\mathrm{e}^{\mathbf{A}^T(t_0-u)}d\tau du\Bigg)\mathrm{e}^{\mathbf{A}^T(t-t_0)} \ . \qquad \text{(B.6)}\end{aligned}$$

From assumption ($\mathbf{x}(t_0)$ and $\mathbf{w}(\tau)$ are independent $\forall\ \tau > t_0$ and $\mathbf{w}(t)$ is a centered white noise), one can state:

- $\mathrm{E}\Big[\mathbf{w}(\tau)\Big(\mathbf{x}(t_0) - \mathbf{m}(t_0)\Big)^T\Big] = \mathbf{0}$

- $\mathrm{E}\left[\mathbf{w}(\tau)\mathbf{w}^T(u)\right] = \mathbf{W}\delta(\tau - u)$.

So:

$$\mathbf{P}(t) = \mathrm{e}^{\mathbf{A}(t-t_0)}\left(\mathbf{P}(t_0) + \int_{t_0}^{t} \mathrm{e}^{\mathbf{A}(t_0-\tau)}\mathbf{M}\,\mathbf{W}\,\mathbf{M}^T\mathrm{e}^{\mathbf{A}^T(t_0-\tau)}d\tau\right)\mathrm{e}^{\mathbf{A}^T(t-t_0)} \ . \tag{B.7}$$

and

$$\dot{\mathbf{P}}(t) = \frac{d\mathbf{P}(t)}{dt} = \mathbf{A}\mathrm{e}^{\mathbf{A}(t-t_0)}\left(\mathbf{P}(t_0) + \cdots\right)\mathrm{e}^{\mathbf{A}^T(t-t_0)} + \mathrm{e}^{\mathbf{A}(t-t_0)}\left(\mathbf{P}(t_0) + \cdots\right)\mathrm{e}^{\mathbf{A}^T(t-t_0)}\mathbf{A}^T$$
$$+\mathrm{e}^{\mathbf{A}(t-t_0)}\left(\mathrm{e}^{\mathbf{A}(t_0-t)}\mathbf{M}\,\mathbf{W}\,\mathbf{M}^T\mathrm{e}^{\mathbf{A}^T(t_0-t)}\right)\mathrm{e}^{\mathbf{A}^T(t-t_0)} \ . \tag{B.8}$$

That is:

$$\boxed{\dot{\mathbf{P}(t)} = \mathbf{A}\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}^T + \mathbf{M}\,\mathbf{W}\,\mathbf{M}^T} \ . \tag{B.9}$$

Remark B.1 *In steady state, the general solution of equation* $\mathbf{AP} + \mathbf{PA}^T + \mathbf{MWM}^T = \mathbf{0}$ *is:*

$$\mathbf{P} = \int_0^\infty \mathrm{e}^{\mathbf{A}t}\mathbf{M}\mathbf{W}\mathbf{M}^T\mathrm{e}^{\mathbf{A}^Tt}dt \ .$$

Indeed:

$$\begin{aligned}
\mathbf{AP} + \mathbf{PA}^T &= \int_0^\infty \mathbf{A}\mathrm{e}^{\mathbf{A}t}\mathbf{M}\mathbf{W}\mathbf{M}^T\mathrm{e}^{\mathbf{A}^Tt} + \mathrm{e}^{\mathbf{A}t}\mathbf{M}\mathbf{W}\mathbf{M}^T\mathrm{e}^{\mathbf{A}^Tt}\mathbf{A}^T dt \\
&= \int_0^\infty \frac{d[\mathrm{e}^{\mathbf{A}t}\mathbf{M}\mathbf{W}\mathbf{M}^T\mathrm{e}^{\mathbf{A}^Tt}]}{dt}dt \\
&= [\mathrm{e}^{\mathbf{A}t}\mathbf{M}\mathbf{W}\mathbf{M}^T\mathrm{e}^{\mathbf{A}^Tt}]_0^\infty = 0 - \mathbf{M}\mathbf{W}\mathbf{M}^T \quad \textit{(iff } \mathbf{A} \textit{ is stable)} \ .
\end{aligned}$$

Let us denote $\mathbf{h}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{M}, \quad \forall t \geq 0$, *the impulse response of the state* $\mathbf{x}$, *one can write:*

$$\begin{aligned}
\mathbf{P} &= \int_0^\infty \mathbf{h}(t)\mathbf{W}\mathbf{h}^T(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{H}(-j\omega)\mathbf{W}\mathbf{H}^T(j\omega)d\omega \quad \textit{(Parseval equality)} \\
&= [\mathcal{L}_{II}^{-1}\mathbf{\Phi}_{xx}(\mathrm{s})]_{s=0} \quad \textit{with:} \quad \mathbf{\Phi}_{xx}(\mathrm{s}) = \mathbf{H}(-\mathrm{s})\mathbf{W}\mathbf{H}^T(\mathrm{s}) \ .
\end{aligned}$$

These last equalities allow to do the link with the following frequency-domain approach (theorem 1.2) and to show that the variance is (with a factor 2π*) the integral of the noise frequency response squared.*

B.2.2 Discrete-time case

Theorem B.1 *Let us consider the discrete-time linear system:*

$$\mathbf{x}(k+1) = \mathbf{A}_d\mathbf{x}(k) + \mathbf{M}_d\mathbf{w}_d(k) \ . \tag{B.10}$$

$\mathbf{w}_d(k)$ *is a pseudo-white gaussian centered noise with a PSD* $\mathbf{W}_d$ *(that is* $\mathrm{E}[\mathbf{w}_d(k)\mathbf{w}_d(k+j)^T] = \mathbf{W}_d\delta_d(j)$*)* . $\mathbf{m}(0)$ *and* $\mathbf{P}(0)$ *denote the mean vector and covariance matrix of initial state* $\mathbf{x}(k_0) = \mathbf{x}(0)$ *(also a random gaussian variable independent of* $\mathbf{w}_d(k)$*). Then* $\mathbf{x}(k)$ *is also a gaussian random signal:*

- *with a mean vector:*

$$\mathbf{m}(k) = \mathrm{E}[\mathbf{x}(k)] = \mathbf{A}_d^{k-k_0}\mathbf{m}(0)$$

- *and a covariance matrix* $\mathbf{P}(k) = \mathrm{E}[(\mathbf{x}(k) - \mathbf{m}(k))(\mathbf{x}(k) - \mathbf{m}(k))^T]$ *solution of the recurrent* LYAPUNOV *equation:*

$$\mathbf{P}(k+1) = \mathbf{A}_d\mathbf{P}(k)\mathbf{A}_d^T + \mathbf{M}_d\mathbf{W}_d\mathbf{M}_d^T \ . \tag{B.11}$$

  If the system is stable (that is all the eigenvalues of $\mathbf{A}_d$ *have a modulus lower than 1) a steady state is reached:* $\mathbf{P}(k+1) = \mathbf{P}(k) = \mathbf{P}$ *is solution of the discrete-time* LYAPUNOV *equation:*

$$\mathbf{P} = \mathbf{A}_d\mathbf{P}\mathbf{A}_d^T + \mathbf{M}_d\mathbf{W}_d\mathbf{M}_d^T \ . \tag{B.12}$$

Proof: (this proof is simpler than in the continuous-time case). In equation (A.12), $\mathbf{x}(k)$ is a linear combination of gaussian random variable. Then one can conclude that $\mathbf{x}(k)$ is a gaussian random variable. $\mathbf{w}_d(k)$ being centered ($\forall k$), one can state that $\mathrm{E}[\mathbf{x}(k)] = \mathbf{A}_d^{k-k_0}\mathbf{m}(0)$.

Lastly, one can note that $\mathbf{x}(k+1) - \mathbf{m}(k+1) = \mathbf{A}_d(\mathbf{x}(k) - \mathbf{m}(k)) + M_d\mathbf{w}_d(k)$. The independence, at each instant k, of **centered** variables $\mathbf{x}(k) - \mathbf{m}(k)$ et $\mathbf{w}_d(k)$ allow us to conclude that $\mathbf{P}(k+1) = \mathbf{A}_d\mathbf{P}(k)\mathbf{A}_d^T + \mathbf{M}_d\mathbf{W}_d\mathbf{M}_d^T$.

Remark B.2 *From equation B.7, one can derive the discrete-time* LYAPUNOV *equation for a continuous system sampled with a sampling period* dt*. It is sufficient to choose* $t_0 = n\,dt$ *et* $t = (n+1)\,dt$ *where* dt *is the sampling period. Then we denote* $\mathbf{P}(t) = \mathbf{P}(n+1)$ *and* $\mathbf{P}(t_0) = \mathbf{P}(n)$*, we have:*

$$\mathbf{P}(n+1) = \mathrm{e}^{\mathbf{A}\,dt}\mathbf{P}(n)\mathrm{e}^{\mathbf{A}^T dt} + \int_0^{dt} \mathrm{e}^{\mathbf{A}u}\mathbf{M}\mathbf{W}\mathbf{M}^T\mathrm{e}^{\mathbf{A}^T u}du$$

Let us remark :

- $\mathrm{e}^{\mathbf{A}dt} = \mathbf{A}_d$*: discrete-time dynamic matrix,*

- $\int_0^{dt} \mathrm{e}^{\mathbf{A}u}\mathbf{M}\mathbf{W}\mathbf{M}^T\mathrm{e}^{\mathbf{A}^T u}du = \mathbf{W}_d$ *is the covariance matrix integrated on a sampling period,*

Then we can write:

$$\boxed{\mathbf{P}_{n+1} = \mathbf{A}_d\,\mathbf{P}_n\,\mathbf{A}_d^T + \mathbf{W}_d}$$

(whose steady state is: $\mathbf{P}_n = \mathbf{A}_d\,\mathbf{P}_n\,\mathbf{A}_d^T + \mathbf{W}_d$*). One find again equation (B.11) with* $\mathbf{M}_d = \mathbf{1}$.

One can verify that :

- *if* $\mathbf{A}$ *is invertible,* $\mathbf{W}_d$ *satisfy the* LYAPUNOV *equation:*

$$\mathbf{A}\mathbf{W}_d + \mathbf{W}_d\mathbf{A}^T + \mathbf{M}\mathbf{W}\mathbf{M}^T - \mathrm{e}^{\mathbf{A}dt}\mathbf{M}\mathbf{W}\mathbf{M}^T\mathrm{e}^{\mathbf{A}^T dt} = 0, \tag{B.13}$$

- $\mathbf{W}_d \approx dt\mathbf{M}\mathbf{W}\mathbf{M}^T$ *if* dt *est small w.r.t. the plant dynamics.*

Example B.1 (Example with Matlab) *Let us consider again exercise 1.3 and complete it by a discrete-time analysis of the variance of the noise* w*:*

```
% Filter definition:
G=tf([-1 100],[1 50 10000])
% State space realization:
[A,B,C,D]=ssdata(G);
% variance determination using continuous-time Lyapunov equation:
P=lyap(A,B*B'); var_w=C*P*C' % ==> var_w=1/50.

% Discrete-time Analysis:
dt=0.001; A_d=expm(A*dt);
Wd=lyap(A,B*B'-A_d*B*B'*A_d');  %In this example: W=I; M=B.
Pd=dlyap(A_d,Wd);var_wd=C*Pd*C'
% ==> We find excatly the variance of w(t): var_w=1/50.

% Approximative calculus using: Wd=dt*B*B'
Pdp=dlyap(A_d,dt*B*B');var_wdp=C*Pdp*C'
% ==> That does not work too bad !!.
```

□

B.3 Frequency-domain approach

B.3.1 Continuous-time case

Theorem 1.2 (rappel). Let us consider a **stable** linear continuous-time system defined by the transfer matrix $\mathbf{G}(\mathrm{s})_{p\times q}$ between the input $\mathbf{w}$ and the output $\mathbf{y}$.

The steady state response $\mathbf{y}$ to a stationary random signal $\mathbf{w}$, characterized by a spectrum $\mathbf{\Phi_{ww}}(\mathrm{s})_{q\times q}$ in the s-plane, is a stationary random signal characterized by a spectrum $\mathbf{\Phi_{yy}}(\mathrm{s})$ in the s-plane such that:

$$\mathbf{\Phi_{yy}}(\mathrm{s})_{p\times p} = \mathbf{G}(-\mathrm{s})\mathbf{\Phi_{ww}}(\mathrm{s})\mathbf{G}^T(\mathrm{s}) \ .$$

Proof: without loss of generality the proof will consider a strictly proper system (without direct feed-through). Let ($\mathbf{A}$, $\mathbf{M}$, $\mathbf{C}$) be a state space realization of the transfer $\mathbf{G}(\mathrm{s})$, i.e.: $\mathbf{G}(\mathrm{s}) = \mathbf{C}(\mathrm{s}\mathbf{1}_n - \mathbf{A})^{-1}\mathbf{M}$.

By definition:

$$\mathbf{\Phi_{yy}}(\mathrm{s}) = \int_{-\infty}^{\infty} \boldsymbol{\phi}_{\mathbf{yy}}(\tau)\mathrm{e}^{-\tau\,s}d\tau = \int_{-\infty}^{\infty} \mathrm{E}\left[\mathbf{y}(t)\mathbf{y}^T(t+\tau)\right]\mathrm{e}^{-\tau\,\mathrm{s}}d\tau \ . \tag{B.14}$$

To compute $\mathbf{y}(t)$ during the steady state, the formulae (A.2) will be used with $\mathbf{x}(t_0) = 0$ and $t_0 = -\infty$:

$$\mathbf{y}(t) = \int_{-\infty}^{t} \mathbf{C}\mathrm{e}^{\mathbf{A}(t-u)}\mathbf{M}\mathbf{w}(u)du = \int_{0}^{\infty} \mathbf{C}\mathrm{e}^{\mathbf{A}v}\mathbf{M}\mathbf{w}(t-v)dv \quad \text{(change of variable: } t-u=v) \ .$$

$$\begin{aligned}
\mathbf{\Phi_{yy}}(\mathrm{s}) &= \int_{-\infty}^{+\infty} \mathrm{E}\left[\int_0^{\infty} \mathbf{C}\mathrm{e}^{\mathbf{A}v}\mathbf{M}\mathbf{w}(t-v)dv \int_0^{\infty} \mathbf{w}^T(t+\tau-u)\mathbf{M}^T\mathrm{e}^{\mathbf{A}^T u}\mathbf{C}^T du\right]\mathrm{e}^{-\tau\mathrm{s}}d\tau \\
&= \int_{-\infty}^{+\infty}\left\{\int\int_0^{\infty} \mathbf{C}\mathrm{e}^{\mathbf{A}v}\mathbf{M}\mathrm{E}\left[w(t-v)w^T(t+\tau-u)\right]\mathbf{M}^T\mathrm{e}^{\mathbf{A}^T u}\mathbf{C}^T\,du\,dv\right\}\mathrm{e}^{-\tau\mathrm{s}}d\tau \\
&= \int_{-\infty}^{+\infty}\left\{\int\int_0^{\infty} \mathbf{C}\mathrm{e}^{\mathbf{A}v}\mathbf{M}\mathrm{e}^{v\,\mathrm{s}}\boldsymbol{\phi}_{\mathbf{ww}}(\tau+v-u)\mathrm{e}^{-(\tau+v-u)\mathrm{s}}\,\mathbf{M}^T\mathrm{e}^{\mathbf{A}^T u}\mathbf{C}^T\mathrm{e}^{-u\mathrm{s}}\,du\,dv\right\}d\tau \\
&= \int_0^{\infty} \mathbf{C}\mathrm{e}^{\mathbf{A}v}\mathbf{M}\mathrm{e}^{v\,\mathrm{s}}\,dv \int_{-\infty}^{+\infty} \boldsymbol{\phi}_{\mathbf{ww}}(\tau+v-u)\mathrm{e}^{-(\tau+v-u)\mathrm{s}}\,d\tau \int_0^{\infty} \mathbf{M}^T\mathrm{e}^{\mathbf{A}^T u}\mathbf{C}^T\mathrm{e}^{-u\mathrm{s}}\,du \\
&= \mathbf{G}(-\mathrm{s})\mathbf{\Phi_{ww}}(\mathrm{s})\mathbf{G}^T(\mathrm{s}) \quad \text{from (A.8) and (B.14)} \ .
\end{aligned}$$

B.3.2 Discrete-time case

Theorem B.2 (Transmission of a noise in a discrete-time linear system) *Let us consider a* ***stable*** *linear discrete-time system defined by the transfer matrix* $\mathbf{G}(\mathrm{z})_{p\times q}$ *between the input* $\mathbf{w}$ *and the output* $\mathbf{y}$. *The steady state response* $\mathbf{y}$ *to a stationary random signal* $\mathbf{w}$, *characterized by a spectrum* $\mathbf{\Phi_{ww}}(\mathrm{z})_{q\times q}$ *in the* z*-plane, is a stationary random signal characterized by a spectrum* $\mathbf{\Phi_{yy}}(\mathrm{z})$ *in the* z*-plane such that:*

$$\mathbf{\Phi_{yy}}(\mathrm{z})_{p\times p} = \mathbf{G}(\mathrm{z}^{-1})\mathbf{\Phi_{ww}}(\mathrm{z})\mathbf{G}^T(\mathrm{z}) \ .$$

Proof: without loss of generality the proof will consider a strictly proposer system (without direct feed-through). Let ($\mathbf{A}$, $\mathbf{M}$, $\mathbf{C}$) be a state space realization of the transfer $\mathbf{G}(\mathrm{s})$, i.e.: $\mathbf{G}(\mathrm{s}) = \mathbf{C}_d(\mathrm{z}\mathbf{1}_n - \mathbf{A}_d)^{-1}\mathbf{M}_d$.

By definition:

$$\mathbf{\Phi_{yy}}(\mathrm{z}) = \sum_{i=-\infty}^{+\infty} \boldsymbol{\phi}_{\mathbf{yy}}(i)\mathrm{z}^{-i} = \sum_{i=-\infty}^{+\infty} \mathrm{E}\left[\mathbf{y}(k)\mathbf{y}^T(k+i)\right]\mathrm{z}^{-i} \tag{B.15}$$

To compute $\mathbf{y}(k)$ during the steady state, the formulae (A.12) will be used with $\mathbf{x}_0 = \mathbf{0}$ and $k_0 = -\infty$:

$$\mathbf{y}(k) = \sum_{j=-\infty}^{k-1} \mathbf{C}_d\mathbf{A}_d^{j-k_0}\mathbf{M}_d\mathbf{w}(k-1-j+k_0) = \sum_{j=0}^{+\infty} \mathbf{C}_d\mathbf{A}_d^{j}\mathbf{M}_d\mathbf{w}(k-1-j) \quad (j \leftarrow j-k_0)\ .$$

$$\begin{aligned}
\mathbf{\Phi_{yy}}(\mathrm{z}) &= \sum_{i=-\infty}^{+\infty} \mathrm{E}\left[\sum_{j=0}^{+\infty} \mathbf{C}_d\mathbf{A}_d^{j}\mathbf{M}_d\mathbf{w}(k-1-j)\ \sum_{l=0}^{+\infty} \mathbf{w}^T(k+i-1-l)\mathbf{M}_d^T\mathbf{A}_d^{T^l}\mathbf{C}_d^T\right]\mathrm{z}^{-i} \\
&= \sum_{i=-\infty}^{+\infty} \left\{\sum_{j=0}^{+\infty}\sum_{l=0}^{+\infty} \mathbf{C}_d\mathbf{A}_d^{j}\mathbf{M}_d\mathrm{E}\left[\mathbf{w}(k-1-j)\mathbf{w}^T(k+i-1-l)\right]\mathbf{M}_d^T\mathbf{A}_d^{T^l}\mathbf{C}_d^T\right\}\mathrm{z}^{-i} \\
&= \sum_{i=-\infty}^{+\infty}\sum_{j=0}^{+\infty}\sum_{l=0}^{+\infty} \mathbf{C}_d\mathbf{A}_d^{j}M_d\mathrm{z}^{j+1}\boldsymbol{\phi}_{\mathbf{ww}}(i-l+j)\mathrm{z}^{-(i-l+j)}\mathbf{M}_d^T\mathbf{A}_d^{T^l}\mathbf{C}_d^T\mathrm{z}^{-l-1} \\
&= \sum_{j=1}^{+\infty} \mathbf{C}_d\mathbf{A}_d^{j-1}\mathbf{M}_d\mathrm{z}^{j} \sum_{k=-\infty}^{+\infty} \boldsymbol{\phi}_{\mathbf{ww}}(k)z^{-k} \sum_{l=1}^{+\infty} \mathbf{M}_d^T\mathbf{A}_d^{T^{l-1}}\mathbf{C}_d^T\mathrm{z}^{-l} \quad (k=i-l+j) \\
&= \mathbf{G}(\mathrm{z}^{-1})\mathbf{\Phi_{ww}}(\mathrm{z})\mathbf{G}^T(\mathrm{z}) \quad \text{from (A.14) and (B.15)}\ .
\end{aligned}$$

Appendix C

Analytical solution of continuous-time differential Riccati equation

C.1 General analytical solution

Let us consider the general differential RICCATI equation (2.14) presented in section 2.2.1 where $\mathbf{P}(t)$ is the unknown $n \times n$ symmetric matrix:

$$\boxed{\dot{\mathbf{P}}(t) = \mathbf{A}\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}^T - \mathbf{P}(t)\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}\mathbf{P}(t) + \mathbf{M}\mathbf{W}\mathbf{M}^T}, \tag{C.1}$$

with the initial condition: $\mathbf{P}(t_0) = \mathbf{P}_0$.

The following procedure to compute $\mathbf{P}(t)$ is derived from chapter 15 of reference [2].

Let us consider the augmented differential equation:

$$\begin{bmatrix} \dot{\mathbf{X}}(t) \\ \dot{\mathbf{Y}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -\mathbf{A}^T & \mathbf{C}^T\mathbf{V}^{-1}\mathbf{C} \\ \mathbf{M}\mathbf{W}\mathbf{M}^T & \mathbf{A} \end{bmatrix}}_{\mathcal{H}} \begin{bmatrix} \mathbf{X}(t) \\ \mathbf{Y}(t) \end{bmatrix} \tag{C.2}$$

with initial conditions [1]: $X(t_0) = \mathbf{1}_n$, $\mathbf{Y}(t_0) = \mathbf{P}_0$.

$\mathcal{H}$ is called the Hamiltonian matrix associated with RICCATI equation.

Then: $\mathbf{P}(t) = \mathbf{Y}(t)\mathbf{X}^{-1}(t)$.

Indeed:

$$\begin{aligned} \dot{\mathbf{P}}(t) &= \dot{\mathbf{Y}}(t)\mathbf{X}^{-1}(t) - \mathbf{Y}(t)\mathbf{X}^{-1}(t)\dot{\mathbf{X}}(t)\mathbf{X}^{-1}(t) \\ &= \mathbf{M}\mathbf{W}\mathbf{M}^T\mathbf{X}(t)\mathbf{X}^{-1}(t) + \mathbf{A}\mathbf{Y}(t)\mathbf{X}^{-1}(t) + \mathbf{Y}(t)\mathbf{X}^{-1}(t)\mathbf{A}^T\mathbf{X}(t)\mathbf{X}^{-1}(t) - \mathbf{Y}(t)\mathbf{X}^{-1}\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}\mathbf{Y}(t)\mathbf{X}^{-1} \\ &= \mathbf{A}\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}^T - \mathbf{P}(t)\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}\mathbf{P}(t) + \mathbf{M}\mathbf{W}\mathbf{M}^T \ . \end{aligned}$$

[1] $\mathbf{1}_n$ is the $n \times n$ identity matrix.

Then:

$$\begin{bmatrix} \mathbf{X}(t) \\ \mathbf{Y}(t) \end{bmatrix} = \mathbf{\Phi}(t_0,t) \begin{bmatrix} \mathbf{1}_n \\ \mathbf{P}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{11}(t_0,t) & \mathbf{\Phi}_{12}(t_0,t) \\ \mathbf{\Phi}_{21}(t_0,t) & \mathbf{\Phi}_{22}(t_0,t) \end{bmatrix} \begin{bmatrix} \mathbf{1}_n \\ \mathbf{P}_0 \end{bmatrix} \tag{C.3}$$

where $\mathbf{\Phi}(t_0,t)$ is the state-transmition matrix of (C.2) (see also: http://en.wikipedia.org/wiki/State-transition_matrix), partionned in four $n \times n$ submatrices $\mathbf{\Phi}_{ij}(t_0,t)$.

Finally the general solution reads:

$$\boxed{\mathbf{P}(t) = [\mathbf{\Phi}_{21}(t_0,t) + \mathbf{\Phi}_{22}(t_0,t)\,\mathbf{P}_0][\mathbf{\Phi}_{11}(t_0,t) + \mathbf{\Phi}_{12}(t_0,t)\,\mathbf{P}_0]^{-1}} \,. \tag{C.4}$$

This analytical solution of $\mathbf{P}(t)$ is still valid in the time-variant case, that is when equation matrix data $\mathbf{A}$, $\mathbf{C}$, $\mathbf{M}$, $\mathbf{W}$ and $\mathbf{V}$ are functions of time t. In the time-invariant case:

$$\mathbf{\Phi}(t_0,t) = \mathrm{e}^{\mathcal{H}(t-t_0)} \,.$$

This matrix exponential can be computed very efficiently using linear algebra library (function `expm` in Matlab).

In the next section, this general solution is used to find the solution presented in the exercice of section 2.3.1.

C.2 Exemple

Let us consider the Riccati differential equation (2.16):

$$\dot{P}(t) = 2aP(t) - P(t)^2 + W \,.$$

The associated hamiltonian matrix reads:

$$\mathcal{H} = \begin{bmatrix} -a & 1 \\ W & a \end{bmatrix} .$$

The matrix exponential $\mathrm{e}^{\mathcal{H}t}$ is computed using an eigen-value/eigen-vector decomposition of matrix $\mathcal{H}$:

$$\mathcal{H} = \mathbf{M}\mathbf{\Lambda}\mathbf{M}^{-1} \quad \Rightarrow \quad \mathrm{e}^{\mathcal{H}t} = \mathbf{M}\mathrm{e}^{\mathbf{\Lambda}t}\mathbf{M}^{-1} = \mathbf{M} \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & \mathrm{e}^{\lambda_2 t} & 0 & \cdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mathrm{e}^{\lambda_n t} \end{bmatrix} \mathbf{M}^{-1} \,,$$

where:

- $\mathbf{\Lambda}$ is the eigenvalue diagonal matrix $\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$,
- $\mathbf{M}$ is the eigenvector matrix $\mathbf{M} = [\mathbf{v}_1, \mathbf{v}_2 \cdots \mathbf{v}_n]$.

Eigen-values:

$\det(\mathrm{s}\mathbf{1}_2 - \mathcal{H}) = \mathrm{s}^2 - a^2 - W = (\mathrm{s}-\lambda)(\mathrm{s}+\lambda)$ with: $\lambda^2 = a^2 + W$ (always > 0 since: $W > 0$) .

Eigen-vectors:

- $\lambda_1 = \lambda$ then $\mathbf{v}_1 / \begin{bmatrix} -a-\lambda & 1 \\ W & a-\lambda \end{bmatrix} \mathbf{v}_1 = \mathbf{0} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ a+\lambda \end{bmatrix}$,
- $\lambda_2 = -\lambda$ then $\mathbf{v}_2 / \begin{bmatrix} -a+\lambda & 1 \\ W & a+\lambda \end{bmatrix} \mathbf{v}_2 = \mathbf{0} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ a-\lambda \end{bmatrix}$,

Then: $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ a+\lambda & a-\lambda \end{bmatrix}$ and $\mathbf{M}^{-1} = \begin{bmatrix} \frac{\lambda-a}{2\lambda} & \frac{1}{2\lambda} \\ \frac{\lambda+a}{2\lambda} & -\frac{1}{2\lambda} \end{bmatrix}$,

and:

$$\begin{aligned} \mathrm{e}^{\mathcal{H}(t-t_0)} &= \begin{bmatrix} 1 & 1 \\ a+\lambda & a-\lambda \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\lambda(t-t_0)} & 0 \\ 0 & \mathrm{e}^{-\lambda(t-t_0)} \end{bmatrix} \begin{bmatrix} \frac{\lambda-a}{2\lambda} & \frac{1}{2\lambda} \\ \frac{\lambda+a}{2\lambda} & -\frac{1}{2\lambda} \end{bmatrix} \\ &= \frac{1}{2\lambda} \begin{bmatrix} \mathrm{e}^{\lambda(t-t_0)}(\lambda-a) + \mathrm{e}^{-\lambda(t-t_0)}(\lambda+a) & \mathrm{e}^{\lambda(t-t_0)} - \mathrm{e}^{-\lambda(t-t_0)} \\ \mathrm{e}^{\lambda(t-t_0)}(\lambda^2-a^2) + \mathrm{e}^{-\lambda(t-t_0)}(a^2-\lambda^2) & \mathrm{e}^{\lambda(t-t_0)}(\lambda+a) + \mathrm{e}^{-\lambda(t-t_0)}(\lambda-a) \end{bmatrix}, \end{aligned}$$

and finally:

$$\begin{aligned} P(t) &= \frac{\mathrm{e}^{\lambda(t-t_0)}(\lambda^2-a^2) + \mathrm{e}^{-\lambda(t-t_0)}(a^2-\lambda^2) + \mathrm{e}^{\lambda(t-t_0)}(\lambda+a)P_0 + \mathrm{e}^{-\lambda(t-t_0)}(\lambda-a)P_0}{\mathrm{e}^{\lambda(t-t_0)}(\lambda-a) + \mathrm{e}^{-\lambda(t-t_0)}(\lambda+a) + \mathrm{e}^{\lambda(t-t_0)}P_0 - \mathrm{e}^{-\lambda(t-t_0)}P_0} \\ &= \frac{\lambda^2 - a^2 + (a+\lambda)P_0 + \mathrm{e}^{-2\lambda(t-t_0)}\left[a^2 - \lambda^2 + (\lambda-a)P_0\right]}{\lambda - a + P_0 + \mathrm{e}^{-2\lambda(t-t_0)}(\lambda+a-P_0)} \end{aligned}$$

Steady state: $\lim_{t\to\infty} P(t) = P_\infty = \frac{\lambda^2 - a^2 + (a+\lambda)P_0}{\lambda - a + P_0} = a + \lambda$

$$\boxed{P_\infty = a + \lambda = a + \sqrt{a^2 + W}}$$ (P_∞ is independant of initial condition P_0)

$P(t)$ can be expressed as a function of P_0, P_∞ and λ (with: $\lambda - a = 2\lambda - P_\infty$):

$$\begin{aligned} P(t) &= \frac{P_\infty(2\lambda - P_\infty) + P_\infty P_0 + \mathrm{e}^{-2\lambda(t-t_0)}(2\lambda - P_\infty)(P_0 - P_\infty)}{2\lambda - P_\infty + P_0 + \mathrm{e}^{-2\lambda(t-t_0)}(P_\infty - P_0)} \\ P(t) &= P_\infty \frac{2\lambda - P_\infty + P_0 + \mathrm{e}^{-2\lambda(t-t_0)}(P_\infty - P_0)\frac{P_\infty - 2\lambda}{P_\infty}}{2\lambda - P_\infty + P_0 + \mathrm{e}^{-2\lambda(t-t_0)}(P_\infty - P_0)} \\ P(t) &= P_\infty + \frac{-2\lambda \mathrm{e}^{-2\lambda(t-t_0)}(P_\infty - P_0)}{2\lambda - P_\infty + P_0 + \mathrm{e}^{-2\lambda(t-t_0)}(P_\infty - P_0)} \end{aligned}$$

$$\boxed{P(t) = P_\infty + \frac{2\lambda(P_0 - P_\infty)}{\mathrm{e}^{2\lambda(t-t_0)}(P_0 - P_\infty + 2\lambda) + P_\infty - P_0}}.$$

One can recognize the solution proposed in equation (2.17) with $k = 2\lambda$.

Appendix D

Matlab demo files

D.1 Function `Kf_t.m`

```
function y=Kf_t(u)
%  y=Kf_t(u)
%   input:
%       * u(1): time (t),
%       * u(2:length(u)): innovation (y(t)-yhat(t))
%   output:
%       * y = Kf(t)*(y(t)-yhat(t))
global a c W P0

% Compute P(t):
Pinf=a+sqrt(a^2+W);
k=2*sqrt(a^2+W);
P=Pinf+k*(P0-Pinf)./(exp(k*u(1))*(P0-Pinf+k)+Pinf-P0);

% Compute Kf(t):
Kf=P*c';

% Output
y=Kf*u(2:length(u));
```

D.2 Script file `demoKalman.m`

```
% Global Variable declaration
global a c W P0
% Kalman model data:
```

```
a=-1;     % state space date
b=1;
c=1;
W=1;   % Process noise spectral density
V=1;
P0=1;    % Initial estimation error variance
X0=20;   % Initial condition for the process output
% Simulation data
dt=0.01;  % Integration sampling period
T=0;      % Time delay in the validation model

% Start simulation:
sim('simuKalman');

% Output plots:
figure plot(output.time,output.signals.values(:,1),'g-') hold on
plot(output.time,output.signals.values(:,2),'k-','LineWidth',2)
plot(output.time,output.signals.values(:,3),'k-')
plot(output.time,output.signals.values(:,4),'r-','LineWidth',2)
% Compute state estimation error variance as a function of time:
t=output.time;
Pinf=a+sqrt(a^2+W);
k=2*sqrt(a^2+W);
Pdet=Pinf+k*(P0-Pinf)./(exp(k*t)*(P0-Pinf+k)+Pinf-P0);
% plot estimation+2*sigma:
plot(output.time,output.signals.values(:,4)+2*sqrt(Pdet),'r-')
% plot estimation-2*sigma:
plot(output.time,output.signals.values(:,4)-2*sqrt(Pdet),'r-')
legend('y(t)','x(t)','u(t)','xhat(t)',...
       'xhat(t)+2\sigma','xhat(t)-2 \sigma')

return

% Solution with a 2nd order Pade approximation of a 1 second delay
% taken into account in the Kalman filter:
T=1; sys=ss(a,b,c,0);
[num,den]=pade(T,2);
ret=tf(num,den);
sysret=sys*ret;
[a,b,c,d]=ssdata(sysret);
W=1;
```

```
[Kf,P]=lqe(a,b,c,W,1);
% Output estimation error variance:
var=c*P*c';

% Start simulation only in steady state behavior:
X0=0;
sim('simuKalman_p'); % same as simuKalman.mdl but
                     % the matlab Function Kf_t is
                     % replaced by a static Gain

% Output plots:
figure plot(output.time,output.signals.values(:,1),'g-') hold on
plot(output.time,output.signals.values(:,2),'k-','LineWidth',2)
plot(output.time,output.signals.values(:,3),'k-')
plot(output.time,output.signals.values(:,4),'r-','LineWidth',2)
% plot estimation+2*sigma:
plot(output.time,output.signals.values(:,4)+2*sqrt(var),'r-')
% plot estimation-2*sigma:
plot(output.time,output.signals.values(:,4)-2*sqrt(var),'r-')
legend('y(t)','x(t)','u(t)','xhat(t)',...
       'xhat(t)+2\sigma','xhat(t)-2 \sigma')
```

D.3 Script file demoKalmand.m

```
% Continuous-time model data:
A=[0 -1;0 0];
B=[1;0];
M=[0;1];
C=[1 0];
D=0;
sysc=ss(A,B,C,D);
V=1;
q=1; W=q^2;
% Continuous-time Kalman filter:
[Kf,P]=lqe(A,M,C,W,V)

dt=0.01;
% Compute Discrete-time model:
sysd=c2d(sysc,dt,'zoh');
[A_d,B_d,C_d,D]=ssdata(sysd)
```

```
Vd=V/dt
Wd=[1/3*q^2*dt^3 -1/2*q^2*dt^2;-1/2*q^2*dt^2 q^2*dt] % Wd can not
%      be computed by a Lyapunov equation (equation B.13) since A
%      is not invertible !! (see how Wd can be computed in LQED)
% Discrete-time Kalman filter using dlqe:
[Kfd1,Pd1]=dlqe(A_d,eye(2),C_d,Wd,Vd)
% Discrete-time Kalman filter using lqed (from continuous-time data):
[Kfd2,Pd2]=lqed(A,M,C,W,V,dt)  %==> same solution !
```