Asymptotic Behavior of Local Times of Compound Poisson Processes with Drift in the Infinite Variance Case

Amaury Lambert · Florian Simatos

Abstract Consider compound Poisson processes with negative drift and no negative jumps, which converge to some spectrally positive Lévy process with nonzero Lévy measure. In this paper, we study the asymptotic behavior of the local time process, in the spatial variable, of these processes killed at two different random times: either at the time of the first visit of the Lévy process to 0, in which case we prove results at the excursion level under suitable conditionings; or at the time when the local time at 0 exceeds some fixed level. We prove that finite-dimensional distributions converge under general assumptions, even if the limiting process is not càdlàg. Making an assumption on the distribution of the jumps of the compound Poisson processes, we strengthen this to get weak convergence. Our assumption allows for the limiting process to be a stable Lévy process with drift. These results have implications on branching processes and in queueing theory, namely, on the scaling limit of binary, homogeneous Crump–Mode–Jagers processes and on the scaling limit of the Processor-Sharing queue length process.

A. Lambert
Laboratoire de Probabilités et Modèles Aléatoires, UMR 7599 CNRS, UPMC Univ Paris 06,
Case courrier 188, 4 Place Jussieu, 75252 Paris Cedex 05, France
e-mail: amaury.lambert@upmc.fr

F. Simatos (✉)
Department of Mathematics and Computer Science, Eindhoven University of Technology,
P.O. Box 513, 5600 MB Eindhoven, The Netherlands
e-mail: f.simatos@tue.nl

1 Introduction

Let X_n be a sequence of spectrally positive compound Poisson processes with drift
which converges weakly to X, then necessarily a spectrally positive Lévy process. The
limiting process has continuous paths as soon as $\mathbb{E}(X_n(1)^2) \to \beta$ for some finite β, a
case henceforth referred to as the finite variance case. In this paper, we focus on the
infinite variance case, when the limiting Lévy process has nonzero Lévy measure. Let
L_n and L be the local time processes associated with X_n and X, respectively, defined
by the occupation density formula

$$\int_0^t f(X_n(s))\mathrm{d}s = \int_{-\infty}^{+\infty} f(x)L_n(x,t)\mathrm{d}x \ \text{ and } \ \int_0^t f(X(s))\mathrm{d}s = \int_{-\infty}^{+\infty} f(x)L(x,t)\mathrm{d}x.$$

Since X_n converges to X, it is natural to also expect L_n to converge to L. Note
however that the map that to a function associates its local time process is not contin-
uous, and so such a result does not automatically follow from the continuous mapping
theorem. In the finite variance case, i.e., when X is Brownian motion, this question
has been looked at in [29] under the assumption that both X_n and X drift to $-\infty$.
Previously, Khoshnevisan [23] investigated this question under different assumptions
on X_n and X but with a different goal, namely to derive convergence rates. The goal of
the present paper is to investigate the asymptotic behavior of L_n in the infinite variance
case, i.e., when the Lévy measure of X is nonzero.

Except for the two aforementioned papers, it seems that this question has not
received much attention. In sharp contrast, there is a rich literature in the closely
related case where X, still a Lévy process, is approximated by a sequence X_n of ran-
dom walks. There are results looking at, e.g., strong and weak invariance principles,
convergence rates, and laws of the iterated logarithm. Nonetheless, the compound
Poisson case in which we will be interested is of practical interest since it finds appli-
cations in the theory of branching processes and in queueing theory (see discussion
below and Sect. 6); besides, the setup that we consider offers specific technical diffi-
culties that do not seem to have been addressed in the random walk case. An overview
of existing results in the random walk case can give insight into the specific technical
difficulties that arise in our framework.

1.1 The Random Walk Case

The most studied case in the random walk case is when X_n is of the form $X_n(t) =
S(nt)/n^{1/2}$ with S a lattice random walk with finite variance, say with step size

distribution ξ, so that X is of the form $\sigma^2 B$ with B a standard Brownian motion. One of the earliest works is in this area was done by Knight [24], see also [5,10,22,33–35,39] for weak convergence results, laws of the iterated logarithm, strong invariance principles, and explicit convergence rates. The introduction of Csörgő and Révész [11] presents a good overview of the literature.

When one drops the finite variance assumption on S (but keeps the lattice assumption), X_n is of the form $X_n(t) = S(nt)/s_n$ for some normalizing sequence (s_n) and X is a stable Lévy process. In this case, significantly fewer results seem available: Borodin [6] has established weak convergence results, Jain and Pruitt [19] a functional law of the iterated logarithm and Kang and Wee [20] L_2-convergence results.

Focusing specifically on weak convergence results, the best results have been obtained by Borodin [5,6], who proved that L_n converges weakly to L if $\mathbb{E}(\xi^2) < +\infty$ (finite variance case) or if ξ is in the domain of attraction of a stable law with index $1 < \alpha < 2$ (infinite variance case).

On the other hand, the picture is far to be as complete in the non-lattice case, even when one only focuses on weak convergence results. First of all, in this case, the very definition of the local time process is subject to discussion, since in contrast with the lattice case, it cannot be defined by keeping track of the number of visits to different points in space. In [11] for instance, five different definitions are discussed. In the finite variance case, Perkins [33] has proved that L_n converges to L, in the sense of finite-dimensional distributions if $\mathbb{E}(\xi^2) < +\infty$, and weakly if $\mathbb{E}(\xi^4) < +\infty$ and $\limsup_{|t|\to\infty} |\mathbb{E}(e^{it\xi})| < 1$; see also [7,11]. In view of the sharp results obtained by Borodin [5] in the lattice case, it is not clear that the conditions derived by Perkins [33] to get weak convergence are optimal. Also, note that this discrepancy, in terms of existing results, between the lattice and non-lattice case, reflects the fact that tightness is significantly more difficult in the non-lattice case. In the non-lattice case, the most involved part of the proof concerns the control of small oscillations of the local time, a difficulty that does not appear in the lattice case, as soon as the amplitude of oscillations is smaller than the lattice mesh (see the discussion after Proposition 7.3).

We finally stress that to our knowledge, the present work is the first study of the asymptotic behavior of L_n in the non-lattice and infinite variance case.

1.2 Main Results

In the present paper, we will be interested in X_n of the form $X_n(t) = Y_n(nt)/s_n$ with (s_n) some normalizing sequence, $Y_n(t) = P_n(t) - t$ and P_n a compound Poisson process whose jump distribution $\xi_n \geq 0$ has infinite second moment. We assume that X_n does not drift to $+\infty$ and that it converges weakly to a spectrally positive Lévy process X. We will focus on the variations in space of the local time processes and consider the asymptotic behavior of the processes $L_n(\,\cdot\,, \tau_n)$ for some specific choices of τ_n. Since $L_n(a, t)$ is increasing in t, this contains the most challenging part of the analysis of local time processes; moreover, this allows for results at the excursion level (see Theorem 2.3). This setup presents two main differences with previous works on random walks.

First, the sequence X_n stems from a *sequence* of compound Poisson processes, when all the aforementioned works in the random walk case consider one random walk S that is scaled. Besides being of practical interest for branching processes and queueing theory, since this allows X to have a drift and, more generally, not to be stable, this variation triggers one important technical difference. Indeed, most of the works on random walks heavily exploit embedding techniques, typically embedding S into X. It is therefore not clear whether such techniques could be adapted to a triangular scheme such as the one considered here.

Second, the image set $\{X_n(t), t \geq 0\}$ is not lattice and X_n has infinite variance; thus, the corresponding random walk counterpart would be the case of non-lattice random walk with infinite variance which, as mentioned previously, has not been treated. Similarly as Perkins [33] in the case of non-lattice random walk with finite variance, we will show that finite-dimensional distributions converge under minimal assumptions and that tightness holds under more stringent ones. However, in contrast with Perkins [33], our assumptions to get tightness will not be in terms of finiteness of some moments but in terms of the specific distribution of ξ_n. In particular, under our assumptions, the limiting process X can be any process of the form $X(t) = Y(t) - dt$ with Y a spectrally positive stable Lévy process with index $1 < \alpha < 2$ and $d \geq 0$.

1.3 Implications

As alluded to above, our results have implications for branching processes and in queueing theory, see Sect. 6 for more details. In short, the process $(L_n(a, \tau_n), a \geq 0)$ for the random times τ_n that will be considered has been shown in [27] to be equal in distribution to a (rescaled) binary, homogeneous Crump–Mode–Jagers (CMJ) branching process. Although the scaling limits of Galton–Watson processes and of Markovian CMJ have been exhaustively studied, see [15,17,31], except for Lambert et al. [29] and Sagitov [37,38], little seems to be known for more general CMJ processes. In particular, we study here for the first time, a sequence of CMJ processes that converges toward a non-Markovian limit process.

Also, CMJ processes are in one-to-one relation with busy cycles of the Processor-Sharing queue via a random time change sometimes called Lamperti transformation in the branching literature. Thus, our results also show that busy cycles of the Processor-Sharing queue converge weakly to excursions that can be explicitly characterized. Leveraging on general results by Lambert and Simatos [28], this implies uniqueness (but not existence) of possible accumulation points of the sequence of queue length processes. This constitutes therefore a major step toward determining the scaling limit (called heavy traffic limit in the queueing literature) of the Processor-Sharing queue in the infinite variance case, which has been a long-standing open question.

1.4 Organization of the Paper

Section 2 sets up the framework of the paper, introduces notation, states assumptions enforced throughout the paper and the two main results. Section 3 is devoted to some preliminary results on Lévy processes. In Sect. 4, we prove that under general

assumptions, finite-dimensional distributions converge while tightness is taken care of under specific technical assumptions in Sect. 5. The (long and tedious) "Appendix" is the most technical part of the paper: It proves the tightness of an auxiliary sequence of processes, which is exploited in Sect. 5 to prove tightness of the processes of interest.

2 Notation and Main Results

2.1 Space $\mathscr{D}$

Let $\mathscr{D}$ be the set of functions $f : [0, \infty) \to \mathbb{R}$ which are right-continuous and have a left limit denoted by $f(t-)$ for any $t > 0$. If f is increasing, we write $f(\infty) = \lim_{x \to +\infty} f(x) \in [0, \infty]$. For $f \in \mathscr{D}$, we define $\underline{f} \in \mathscr{D}$ the function f reflected above its past infimum through the following formula:

$$\underline{f}(t) = f(t) - \min \left(0, \inf_{0 \leq s \leq t} f(s) \right).$$

For $f_n, f \in \mathscr{D}$, we note $f_n \to f$ for convergence in the Skorohod J_1 topology (see for instance [4] or Chapter VI in [18]). For any function $f \in \mathscr{D}$, we introduce the family of mappings $(T_f(A, k), A \subset \mathbb{R}, k \geq 0)$ defined recursively for any subset $A \subset \mathbb{R}$ by $T_f(A, 0) = 0$ and for $k \geq 1$,

$$T_f(A, k) = \inf \left\{ t > T_f(A, k - 1) : f(t) \in A \text{ or } f(t-) \in A \right\}.$$

We will write for simplicity $T_f(A) = T_f(A, 1)$ and when $A = \{a\}$ is a singleton, we will write $T_f(a, k)$ and $T_f(a)$ in place of $T_f(A, k)$ and $T_f(A)$, respectively. A function $f \in \mathscr{D}$ is called an excursion if $T_f(0) = +\infty$, or if $T_f(0) \in (0, +\infty)$ and $f(t) = 0$ for all $t \geq T_f(0)$. By $\mathscr{E} \subset \mathscr{D}$, we will denote the set of excursions.

We use the canonical notation for càdlàg stochastic processes. Let $\Omega = \mathscr{D}$ and $X = (X(t), t \geq 0)$ be the coordinate process, defined by $X(t) = X_\omega(t) = \omega(t)$. We will systematically omit the argument of functional operators when they are applied at X; $T(A, k)$ for instance stands for the random variable $T_X(A, k)$. Finally, let $X^0 = X(\cdot \wedge T(0))$ be the process X stopped upon its first visit to 0, with $X^0 = X$ when $T(0) = +\infty$.

2.2 Sequence of Lévy Processes

For $n \geq 1$, fix $\kappa_n \in (0, +\infty)$, and Λ_n some positive random variable. For $x \in \mathbb{R}$, let $\mathbb{P}_n^x$ be the law of a Lévy process started at x with Laplace exponent ψ_n given by

$$\psi_n(\lambda) = \lambda - \kappa_n \mathbb{E} \left(1 - e^{-\lambda \Lambda_n} \right), \quad \lambda \geq 0.$$

Noting $\pi_n(\mathrm{d}a) = \kappa_n \mathbb{P}(\Lambda_n \in \mathrm{d}a)$, one sees that X under $\mathbb{P}_n^0$ is of the form $P_n(t) - t$ with P_n a compound Poisson process with Lévy measure π_n. If we denote by η_n the largest root of the convex mapping ψ_n, then ψ_n is increasing in $[\eta_n, +\infty)$ and its

inverse is denoted by ϕ_n. In particular, $\phi_n(0) = \eta_n$, which equals zero as soon as $\psi_n'(0+) \geq 0$.

Let Λ_n^* be the forward recurrence time of Λ_n, also called size-biased distribution, which is the random variable with density $\mathbb{P}(\Lambda_n \geq x)/\mathbb{E}(\Lambda_n)$ with respect to Lebesgue measure. Let $\mathbb{P}_n$ and $\mathbb{P}_n^*$ be the measures defined by $\mathbb{P}_n(\cdot) = \int \mathbb{P}_n^x(\cdot)\mathbb{P}(\Lambda_n \in \mathrm{d}x)$ and $\mathbb{P}_n^*(\cdot) = \int \mathbb{P}_n^x(\cdot)\mathbb{P}(\Lambda_n^* \in \mathrm{d}x)$. We will use repeatedly the following result, which characterizes the law of the overshoot of X under $\mathbb{P}_n^0$ when X under $\mathbb{P}_n^0$ does not drift to $+\infty$ (an assumption that will be enforced throughout the paper).

Lemma 2.1 (Theorem VII.17 in [2]) *If X under $\mathbb{P}_n^0$ does not drift to $+\infty$, then $X\big(T((0,\infty))\big)$ under $\mathbb{P}_n^0(\cdot \mid T((0,\infty)) < +\infty)$ is equal in distribution to Λ_n^*.*

Let w_n be the scale function of X under $\mathbb{P}_n^0$, which is the only absolutely continuous increasing function with Laplace transform

$$\int\limits_0^\infty e^{-\lambda x} w_n(x)\mathrm{d}x = \frac{1}{\psi_n(\lambda)}, \ \lambda > \phi_n(0).$$

It is well known and can be easily computed that $w_n(0) = \lim_{\lambda\to\infty}(\lambda/\psi_n(\lambda)) = 1$. Scale functions play a central role with regard to exit problems, see forthcoming formula (1). We now define the sequence of rescaled processes that will be the main focus of the paper.

Fix from now on some sequence (s_n) of strictly positive real numbers, which increases to infinity, and for $n \geq 1$ define $r_n = n/s_n$. Let $\mathbf{P}_n^x$, $\mathbf{P}_n$ and $\mathbf{P}_n^*$ be the laws of $X(nt)/s_n$ under $\mathbb{P}_n^{xs_n}$, $\mathbb{P}_n$ and $\mathbb{P}_n^*$, respectively, and let $\underline{\mathbf{P}}_n^0$ be the law of $\underline{X}$ under $\mathbf{P}_n^0$. Then, $\mathbf{P}_n^a$ is the law of a Lévy process started at a, with Lévy exponent $\Psi_n(\lambda) = n\psi_n(\lambda/s_n)$, Lévy measure $\Pi_n(\mathrm{d}a) = n\kappa_n\mathbb{P}(\Lambda_n/s_n \in \mathrm{d}a)$ and scale function $W_n(a) = w_n(as_n)/r_n$. Set also $\Phi_n(\lambda) = s_n\phi_n(\lambda/n)$ so that $\Phi_n(0)$ is the largest root of Ψ_n and Φ_n is the inverse of Ψ_n on $[\Phi_n(0), +\infty)$. Throughout the paper, we use $\Rightarrow$ to denote weak convergence.

Convergence of Lévy Process *In the rest of the paper, we consider $\mathbf{P}^0$ the law of a spectrally positive Lévy process with infinite variation and nonzero Lévy measure started at 0. It is assumed throughout that (1) for each $n \geq 1$, X under $\mathbb{P}_n^0$ does not drift to $+\infty$ and (2) $\mathbf{P}_n^0 \Rightarrow \mathbf{P}^0$.*

We also define Ψ the Lévy exponent and W the scale function associated with $\mathbf{P}^0$, as well as $\underline{\mathbf{P}}^0$ the law of $\underline{X}$ under $\mathbf{P}^0$. The previous assumptions have two immediate consequences: (1) X under $\mathbf{P}^0$ does not drift to $+\infty$; in particular, Ψ is increasing and letting Φ be its inverse, it is not hard to show that $\Phi_n \to \Phi$ and (2) $\kappa_n\mathbb{E}(\Lambda_n) \leq 1$ and $\kappa_n\mathbb{E}(\Lambda_n) \to 1$; in particular, $\mathbb{P}_n^0$ is close to the law of a critical Lévy process.

As alluded to above, scale functions play a central role with regard to exit problems. This comes from the following relation, that holds for any $0 \leq a < b \leq \infty$, see for instance Theorem VII.8 in [2]:

$$\mathbf{P}_n^0\left(T(-(b-a)) < T(a)\right) = \frac{W_n(a)}{W_n(b)} \text{ and } \mathbf{P}^0\left(T(-(b-a)) < T(a)\right) = \frac{W(a)}{W(b)}.$$

(1)

2.3 Local Times and Excursion Measures

For a given function $f \in \mathscr{D}$, let $\mu_{t,f}$ for any $t \geq 0$ be its occupation measure defined via

$$\int_{\mathbb{R}} \varphi(a)\mu_{t,f}(\mathrm{d}a) = \int_0^t \varphi(f(s))\mathrm{d}s$$

for every measurable function $\varphi \geq 0$. When $\mu_{t,f}$ is absolutely continuous with respect to Lebesgue measure, we denote by $L_f(\cdot, t)$ its Radon–Nikodym derivative restricted to $[0, \infty)$, satisfying the so-called occupation density formula

$$\int_0^t \varphi(f(s))\mathrm{d}s = \int_0^\infty \varphi(a)L_f(a, t)\mathrm{d}a$$

for any $t \geq 0$ and any measurable function $\varphi \geq 0$ with $\varphi(a) = 0$ for $a < 0$. The functional $L_f(\cdot, \cdot)$ is known as the local time process of f and is uniquely determined up to sets of zero Lebesgue measure (in the space variable). Let $T_f^L(\zeta)$ for $\zeta \geq 0$ be the first time the amount of local time accumulated at level 0 exceeds ζ:

$$T_f^L(\zeta) = \inf\left\{t \geq 0 : L_f(0, t) \geq \zeta\right\}.$$

In the sequel, we will be interested in the local time process $L = L_X$ under various measures. Under $\mathbf{P}_n^0$, L is simply given by

$$L(a, t) = \frac{1}{r_n} \sum_{0 \leq s \leq t} \mathbb{1}_{\{X(s)=a\}}, \ t, a \geq 0, \ \mathbf{P}_n^0\text{-almost surely.}$$

Further, it is known that L under $\mathbf{P}^0$ is almost surely well defined, see for instance [2]. We will consider $\mathscr{N}$ the excursion measure of X under $\mathbf{P}^0$ associated with the local time $(L(0, t), t \geq 0)$, and $\underline{\mathscr{N}}$ the excursion measure of $\underline{X}$ under $\mathbf{P}^0$ normalized by considering the local time at 0 of $\underline{X}$ equal to $(\min(0, \inf_{[0,t]} X), t \geq 0)$. Under $\mathscr{N}$ and $\underline{\mathscr{N}}$, we will consider the process L^0 defined as follows:

$$L^0 = \left(L(a, T(0)), a \geq 0\right).$$

This process indeed is well defined on $(0, \infty)$ (its value at 0 is zero) under $\mathscr{N}$ and $\underline{\mathscr{N}}$, since $\mathscr{N}$ and $\underline{\mathscr{N}}$ both have the same semigroup in $(0, \infty)$ as the Lévy process under $(\mathbf{P}^a, a > 0)$ killed upon reaching zero. Similarly, the process $L_{\underline{X}}$ is well defined

under $\mathbf{P}^0$, because on $[\varepsilon, \infty)$, it can be expressed as a finite sum of the local time processes of independent excursions distributed as $\underline{\mathscr{N}}(\,\cdot\,|\, \sup X > \varepsilon)$. Recall from the beginning of the paragraph that the normalization at 0 of L_X is slightly different, so that the occupation formula for this local time only holds on $(0, \infty)$.

2.4 Additional Assumption for Tightness

It will be shown that the mere assumption $\mathbf{P}_n^0 \Rightarrow \mathbf{P}^0$ implies that the finite-dimensional distributions of L under $\mathbf{P}_n^0$ converge toward those of L under $\mathbf{P}^0$ (see forthcoming Theorems 2.3 and 2.4 for precise statements). However, it is known that the local time process of a Lévy process is either jointly continuous or has a very wild behavior, see [1] for a general criterion. In the latter case, for every $t, \varepsilon > 0$ the set $\{L(a, t), a \in \mathbb{Q} \cap (-\varepsilon, \varepsilon)\}$ is almost surely dense in $[0, \infty)$. When focusing on spectrally positive Lévy processes with infinite variation, Barlow's criterion, which is in general difficult to determine, takes the following simple form.

Lemma 2.2 *The local time process of X under $\mathbf{P}^0$ is jointly continuous if and only if*

$$\int^{\infty} \frac{\mathrm{d}\lambda}{\Psi(\lambda)\sqrt{\log \lambda}} < +\infty. \tag{2}$$

Proof It will be proved in Lemma 3.6 that $\mathbf{E}^0(L(0, T(-a)) = W(a)$ for any $a > 0$. Then, Lemma V.11 and Theorem V.15 in [2] ensure that L under $\mathbf{P}^0$ is jointly continuous if and only if

$$\int_0 \sqrt{\log(1/W^{-1}(x))}\mathrm{d}x < +\infty.$$

Using the change of variables $y = W^{-1}(x)$ and integrating by parts, it can be shown that

$$\int_0^{W(1)} \sqrt{\log(1/W^{-1}(x))}\mathrm{d}x = \frac{1}{2}\int_0^1 \frac{W(u)}{u\sqrt{\log(1/u)}}\mathrm{d}u.$$

Since there exist two finite constants $0 < c \le C$ such that $c/(u\Psi(1/u)) \le W(u) \le C/(u\Psi(1/u))$ for every $u \ge 0$ (see Proposition III.1 or the proof of Proposition VII.10 in [2]), we obtain

$$\int_0 \sqrt{\log(1/W^{-1}(x))}\mathrm{d}x < +\infty \Longleftrightarrow \int_0 \frac{\mathrm{d}u}{u^2\Psi(1/u)\sqrt{\log(1/u)}} < +\infty$$

and the change of variables $\lambda = 1/u$ in the last integral therefore gives the result. $\quad\square$

In particular, when (2) fails, L under $\mathbf{P}^0$ is not even càdlàg (in the spatial variable) and so cannot be the weak limit of any sequence, when using Skorohod's topology. It is tempting to think that (2) is enough for L under $\mathbf{P}_n^0$ to converge weakly toward L under $\mathbf{P}^0$, and we leave this question open for future research. In the present paper, we will prove tightness (and hence weak convergence) under the following assumption.

Tightness Assumption *In the rest of the paper, we fix some $1 < \alpha < 2$ and denote by Λ the random variable with tail distribution function $\mathbb{P}(\Lambda \geq s) = (1 + s)^{-\alpha}$. We will say that the tightness assumption holds if for $n \geq 1$ we have $s_n = n^{1/\alpha}$ and $\Lambda_n = \Lambda$.*

Note that under this assumption, $\mathbf{P}^0$ is the law of a Lévy process of the form $Y(t) - dt$ with Y a stable Lévy process with index α and $d \geq 0$. It is then not difficult to check that (2) is satisfied and so our limiting processes will be continuous. However, we will show weak convergence without the *a priori* knowledge given to us by Lemma 2.2 that the limiting process is continuous. But since our pre-limit processes make deterministic jumps of size $1/r_n \to 0$, it follows from our approach that $(L(a, T), a \geq 0)$ is continuous for some specific random times T, thus proving directly (without resorting to Barlow's more general and complete result) that the local time process of X is continuous.

2.5 Main Results

The following two theorems are the main results of the paper.

Theorem 2.3 *For any $a_0 > 0$, the two sequences of processes L^0 under $\mathbf{P}_n^*(\,\cdot\,|\,T(a_0) < T(0))$ and $\mathbf{P}_n(\,\cdot\,|\,T(a_0) < T(0))$ converge in the sense of finite-dimensional distributions to L^0 under $\mathscr{N}(\,\cdot\,|\,T(a_0) < T(0))$ and $\underline{\mathscr{N}}(\,\cdot\,|\,T(a_0) < T(0))$, respectively. If in addition, the tightness assumption holds, then both convergences hold in the sense of weak convergence.*

When reading the following theorem, it is useful to have in mind that $r_n \to +\infty$, since $r_n = 1/W_n(0)$ and it will be proved in Lemma 3.4 that $W_n(0) \to W(0) = 0$.

Theorem 2.4 *Let $\zeta > 0$ and (z_n) be any integer sequence such that $\zeta_n = z_n/r_n \to \zeta$. Then, the sequence of processes $L(\,\cdot\,, T^L(\zeta_n))$ under $\mathbf{P}_n^0(\,\cdot\,|\,T^L(\zeta_n) < +\infty)$ converges in the sense of finite-dimensional distributions to $L(\,\cdot\,, T^L(\zeta))$ under $\mathbf{P}^0(\,\cdot\,|\,T^L(\zeta) < +\infty)$. If in addition, the tightness assumption holds, then the convergence holds in the sense of weak convergence.*

Note that since X under $\mathbf{P}^0$ is assumed to have discontinuous paths, Theorem 1.1 in [13] guarantees that the limiting process in Theorem 2.4 is not Markovian. Decomposing the path of X into its excursions away from 0, it could be also be shown that L^0 under $\mathscr{N}$ and $\underline{\mathscr{N}}$ does not satisfy the Markov property.

As a last remark, we stress that it is possible to extend the proof of Theorem 2.4 to get convergence of the processes $(L(a, T^L(\zeta_n)), a \in \mathbb{R})$ instead of $(L(a, T^L(\zeta_n)), a \geq 0)$. Both the proofs of convergence of the finite-dimensional distributions and of tightness can be adapted to this case with no major changes, though at the expense of more computations.

3 Preliminary Results

We prove in this section preliminary results that will be used several times in the paper. We first need results concerning the continuity of hitting times, cf. [18, Proposition VI.2.11] for closely related results.

Lemma 3.1 *Let f_n, $f \in \mathcal{D}$ such that $f_n \to f$ and A be any finite subset of $\mathbb{R}$. Assume that f_n for each $n \geq 1$ has no negative jumps and that:*

  (i) $f(0) \notin A$;
  (ii) $T_f(A)$ *is finite;*
  (iii) f *has no negative jumps;*
  (iv) for any $a \in A$ *and* $\varepsilon > 0$, $\sup_{[T_f(a), T_f(a)+\varepsilon]} f > a$ *and* $\inf_{[T_f(a), T_f(a)+\varepsilon]} f < a$;

Then, $T_{f_n}(A) \to T_f(A)$. In particular, if in addition to (i)–(iv) above f also satisfies the following condition:

(v) f *is continuous at* $T_f(a)$ *for each* $a \in A$;

then $f_n(T_{f_n}(A)) \to f(T_f(A))$, with $f_n(T_{f_n}(A))$ being well defined for n large enough.

Proof In the rest of the proof, let $A = \{a_k, 1 \leq k \leq K\}$, $T = T_f(A)$ and $T_n = T_{f_n}(A)$. Assume that the result holds for $K = 1$: Then, for each $a \in A$, one has $T_{f_n}(a) \to T_f(a)$. Since $T = \min_{a \in A} T_f(a)$ and $T_n = \min_{a \in A} T_{f_n}(a)$, one deduces that $T_n \to T$. Thus, the result only needs to be proved for $K = 1$, which we assume from now on. We then write for simplicity $a = a_1$.

We first prove that $\liminf_n T_n \geq T$. Let $\varepsilon > 0$ and $t \in (T - \varepsilon, T)$ be such that f is continuous at t: then

$$\lim_{n \to +\infty} \inf_{[0,t]} |f_n - a| = \inf_{[0,t]} |f - a|.$$

Since $f(0) \neq a$ and $t < T$, we get $\inf_{[0,t]} |f - a| > 0$ and so $\inf_{[0,t]} |f_n - a| > 0$ for n large enough. For those n, we therefore have $T_n \geq t$ and so $\liminf_n T_n \geq t$. Since $t \geq T - \varepsilon$ and ε is arbitrary, letting $\varepsilon \to 0$ gives $\liminf_n T_n \geq T$.

We now prove that $\limsup_n T_n \leq T$. Fix $\varepsilon > 0$ and let $t_2 \in (T, T+\varepsilon)$ be a continuity point of f such that $f(t_2) < a$ and $t_1 \in (T, t_2)$ another continuity point of f such that $f(t_1) > a$. Since t_1 and t_2 are continuity points for f, we have $f_n(t_i) \to f(t_i)$ for $i = 1, 2$. Since $f(t_1) > a > f(t_2)$, there exists $n_0 \geq 0$ such that $f_n(t_1) > a > f_n(t_2)$ for all $n \geq n_0$. Since f_n has no negative jumps, for those n, there necessarily exists $t_3 \in (t_1, t_2)$ such that $f_n(t_3) = a$ which implies $T_n \leq t_3$. Since $t_3 \leq t_2 \leq T + \varepsilon$, we obtain $T_n \leq T + \varepsilon$ for all $n \geq n_0$ and in particular $\limsup_n T_n \leq T + \varepsilon$. Letting $\varepsilon \to 0$ achieves the proof. □

Lemma 3.2 *For any finite subset $A \subset \mathbb{R} \setminus \{0\}$, $(T(A), X(T(A)))$ under $\mathbf{P}_n^0(\cdot \mid T(A) < +\infty)$ converges weakly to $(T(A), X(T(A)))$ under $\mathbf{P}^0(\cdot \mid T(A) < +\infty)$. If in addition $\min A > 0$, then $(T(A), X(T(A)))$ under $\underline{\mathbf{P}}_n^0$ converges weakly to $(T(A), X(T(A)))$ under $\underline{\mathbf{P}}^0$.*

Proof In the rest of the proof, let $H_A \subset \mathscr{D}$ be the set of functions $f \in \mathscr{D}$ satisfying the five conditions (i)–(v) of Lemma 3.1. Let us first assume that min $A > 0$ and show the convergence under $\underline{\mathbf{P}}_n^0$. By assumption, we have $\mathbf{P}_n^0 \Rightarrow \mathbf{P}^0$ and so the continuous mapping theorem implies that $\underline{\mathbf{P}}_n^0 \Rightarrow \underline{\mathbf{P}}^0$. Moreover, since X under $\mathbf{P}^0$ has by assumption infinite variation, one can easily check that $\underline{\mathbf{P}}^0(H_A) = 1$ and so the continuous mapping theorem together with Lemma 3.1 gives the result for $\underline{\mathbf{P}}^0$.

Let us now show the result under $\mathbf{P}^0(\cdot \mid T(A) < +\infty)$, so we do not assume anymore min $A > 0$. Since $\mathbf{P}^0(H_A) = 1$, using the same arguments as under $\underline{\mathbf{P}}_n^0$, one sees that it is enough to prove that $\mathbf{P}_n^0(\cdot \mid T(A) < +\infty) \Rightarrow \mathbf{P}^0(\cdot \mid T(A) < +\infty)$. If min $A \leq 0$ or if all the Lévy processes are critical, then $\mathbf{P}_n^0(T(A) < +\infty) = \mathbf{P}^0(T(A) < +\infty) = 1$ and so this last convergence is the same as $\mathbf{P}_n^0 \Rightarrow \mathbf{P}^0$. Otherwise, since $\{T(A) < +\infty\} = \{\sup X \geq \min A\}$, it is sufficient to check that $\mathbf{P}^0(\sup X = a) = 0$ for all a and to prove that $(X, \sup X)$ under $\mathbf{P}_n^0$ converges to $(X, \sup X)$ under $\mathbf{P}^0$, which we do now.

Taking $b = \infty$ in (1) shows that

$$\mathbf{P}_n^0(\sup X < a) = \frac{W_n(a)}{W_n(\infty)} \quad \text{and} \quad \mathbf{P}^0(\sup X < a) = \frac{W(a)}{W(\infty)},$$

so that indeed $\mathbf{P}^0(\sup X = a) = 0$ (W is continuous), and the laws of $\sup X$ converge (by Lemma 3.4).

As a consequence, the laws of $(X, \sup X)$ are tight. Let (Y, M) be any accumulation point of this sequence. Then, Y must be equal in distribution to X under $\mathbf{P}^0$ and M must be equal in distribution to $\sup X$ under $\mathbf{P}^0$. As a consequence, M and $\sup Y$ have the same distribution, but (Y, M) does not necessarily have the same law as $(X, \sup X)$. To prove this, it is sufficient to show that $M = \sup Y$. By Skorokhod embedding theorem, we can find a sequence (Y_n, M_n) defined on the same probability space as (Y, M) and converging almost surely to (Y, M), such that for each n, (Y_n, M_n) has the law of $(X, \sup X)$ under $\mathbf{P}_n^0$. Then, for any continuity point t of Y, $\sup_{[0,t]} Y_n$ converges to $\sup_{[0,t]} Y$. This shows that $M \geq \sup_{[0,t]} Y$, and since continuity points are dense, $M \geq \sup Y$. Now since M and $\sup Y$ have the same distribution, the almost sure inequality $M \geq \sup Y$ actually is an almost sure equality, hence the result. □

Lemma 3.3 *For any $a > 0$,*

$$\lim_{n \to +\infty} \kappa_n s_n \mathbf{P}_n(T(a) < T(0)) = \mathscr{N}(T(a) < T(0)).$$

Proof Let $a > 0$ and $S = \sup_{[0, T(-1)]} \underline{X}$: The exponential formula for the Poisson point process of excursions gives

$$\mathbf{P}_n^0(S < a) = \exp\left(-\kappa_n s_n \mathbf{P}_n(T(a) < T(0))\right) \quad \text{and}$$
$$\mathbf{P}^0(S < a) = \exp\left(-\mathscr{N}(T(a) < T(0))\right).$$

On the other hand, using Lemma 3.1 and continuity properties of the sup operator, it is not hard to see that S under $\mathbf{P}_n^0$ converges weakly to S under $\mathbf{P}^0$. Since the distribution of S under $\mathbf{P}^0$ has not atom, we get $\mathbf{P}_n^0 (S < a) \to \mathbf{P}^0 (S < a)$ which, in view of the last display, concludes the proof. ☐

It is well known that scale functions are everywhere differentiable, which justifies the following statement.

Lemma 3.4 *For every $a \geq 0$, it holds that $W_n(a) \to W(a)$ and $W_n'(a) \to W'(a)$. Moreover, $W_n(\infty) \to W(\infty)$.*

Proof First observe that the pointwise convergence of $1/\Psi_n$ to $1/\Psi$, which are the respective Laplace transforms of W_n and W, along with Theorem XIII.1.2 in [14] and the continuity of W ensures that $W_n(a) \to W(a)$ for any $a \geq 0$.

On the other hand, it is elementary to show that

$$\mathbf{P}_n (T(a) < T(0)) = \frac{W_n'(a)}{\kappa_n s_n W_n(a)},$$

and similarly, one can obtain $\underline{\mathcal{N}}(T(a) < T(0)) = W'(a)/W(a)$ as follows (which will prove $W_n'(a) \to W'(a)$ in view of Lemma 3.3). Setting $h(x) := \underline{\mathcal{N}}(T(x) < T(0))$, by the exponential formula for the Poisson point process of excursions, we get, by (1), for any $0 < a < b$,

$$\frac{W(a)}{W(b)} = \mathbf{P}^0 (T(-(b-a)) < T(a))$$

$$= \exp\left(- \int_0^{b-a} \underline{\mathcal{N}}(T(a+x) < T(0)) \, \mathrm{d}x \right)$$

$$= \exp\left(- \int_a^b h(u) \, \mathrm{d}u \right).$$

This entails $h(a) = W'(a)/W(a)$, for example, by differentiating the last equality. The last convergence comes from the relations $W_n(\infty) = 1/\Psi_n'(0+)$ and $W(\infty) = 1/\Psi'(0+)$. ☐

Lemma 3.5 *For any $\varepsilon > 0$,*

$$\lim_{n \to +\infty} \kappa_n s_n \mathbf{P}_n (T(0) > \varepsilon) = \underline{\mathcal{N}}(T(0) > \varepsilon).$$

Proof We abbreviate $T(0)$ into T. According to Chapter VII in [2], under $\mathbf{P}^0$ (resp. $\mathbf{P}_n^0$), the first passage time process of X in the negative half line is a subordinator with Laplace exponent Φ (resp. Φ_n). This has two consequences.

The first one, obtained by considering X under $\mathbf{P}^0$, is that $\Phi(\lambda) = d\lambda + \underline{\mathcal{N}}(1 - e^{-\lambda T})$, where $d \geq 0$ is a drift coefficient. Actually, $d = 0$, since the Lévy process X

under $\mathbf{P}^0$ has infinite variation, so that $\lim_{\lambda \to \infty} \lambda / \Psi(\lambda) = 0 = \lim_{\lambda \to \infty} \Phi(\lambda)/\lambda = d$. We thus have $\Phi(\lambda) = \underline{\mathcal{N}}(1 - e^{-\lambda T})$. The second one, obtained by considering X under $\mathbf{P}_n^0$, is that $\mathbf{E}_n \left(e^{-\lambda T} \right) = \mathbb{E} \left(e^{-\Phi_n(\lambda)\Lambda_n/s_n} \right)$, so that

$$\mathbf{E}_n \left(1 - e^{-\lambda T} \right) = \frac{(n/s_n)\Phi_n(\lambda) - \Psi_n(\Phi_n(\lambda))}{n\kappa_n} = \frac{\Phi_n(\lambda)}{\kappa_n s_n} - \frac{\lambda}{n\kappa_n}.$$

Multiplying each side with $\kappa_n s_n$, letting $n \to +\infty$ and using that $\Phi_n(\lambda) \to \Phi(\lambda)$, this shows that $\kappa_n s_n \mathbf{E}_n(1 - e^{-\lambda T})$ converges to $\underline{\mathcal{N}}(1 - e^{-\lambda T})$, i.e.,

$$\lim_{n \to \infty} \int_0^\infty \mathrm{d}x \, e^{-\lambda x} \kappa_n s_n \mathbf{P}_n(T > x) = \int_0^\infty \mathrm{d}x \, e^{-\lambda x} \underline{\mathcal{N}}(T > x).$$

Then, Theorem XIII.1.2 in [14] implies that $\kappa_n s_n \mathbf{P}_n(T > x) \to \underline{\mathcal{N}}(T > x)$ for any $x > 0$ such that $\underline{\mathcal{N}}(T = x) = 0$, that is, for any $x > 0$. □

Lemma 3.6 *Recall that $r_n = n/s_n$. If A is a finite subset of $\mathbb{R}$ such that $0 \notin A$, then*

$$\lim_{n \to +\infty} r_n \mathbf{P}_n^0 (T(A) < T(0)) = \mathcal{N}(T(A) < T(0)).$$

Proof Since X under $\mathbf{P}_n^0$ is spectrally positive, the following simplifications occur:

- if $\min A > 0$ then $\mathbf{P}_n^0 (T(A) < T(0)) = \mathbf{P}_n^0 (T(\min A) < T(0))$;
- if $\max A < 0$ then $\mathbf{P}_n^0 (T(A) < T(0)) = \mathbf{P}_n^0 (T(\max A) < T(0))$;
- if $\min A < 0 < \max A$ then $\mathbf{P}_n^0 (T(A) < T(0)) = \mathbf{P}_n^0 (T(\min A) \wedge T(\max A) < T(0))$.

Thus, to prove the result, there are only three cases to consider: $A = \{a\}$ with $a > 0$, $A = \{a\}$ with $a < 0$ or $A = \{a, b\}$ with $a < 0 < b$.

First case: $A = \{a\}$ with $a < 0$. For any $x > 0$, $L(x, T(a))$ under $\mathbf{P}^0$ is 0 if $T(a) < T(x)$ and otherwise it is exponentially distributed with parameter $\mathcal{N}(T(a - x) < T(0))$, so we get $v_a(x) := \mathbf{E}^0(L(x), T(a)) = \mathbf{P}^0(T(x) < T(a))/\mathcal{N}(T(a - x) < T(0))$. Now, since $\mathbf{P}^0$ only charges Lévy processes with infinite variation, the map $x \mapsto T(x)$ is $\mathbf{P}^0$-a.s. and $\mathcal{N}$-a.e. continuous, so by monotone convergence, v_a is right-continuous and $v_a(0) = 1/\mathcal{N}(T(a) < T(0))$. Now, we refer the reader to, e.g., the second display on page 207 of [3], to check that the Lévy process with law $\mathbf{P}^0$ and killed upon hitting a has a continuous potential density, say u_a, whose value at 0 is equal to $W(-a)$. In addition, by the occupation density formula and Fubini–Tonelli theorem, for any non-negative function φ vanishing on the negative half line, $\mathbf{E}^0 \int_0^{T(a)} \varphi(X(s))\mathrm{d}s = \int_0^\infty \varphi(x)v_a(x)\mathrm{d}x$, which, by definition of u_a, also equals $\int_0^\infty \varphi(x)u_a(x)\mathrm{d}x$. Since u_a is continuous and v_a is right-continuous, u_a and v_a are equal, and in particular $v_a(0) = 1/\mathcal{N}(T(a) < T(0)) = u_a(0) = W(-a)$.

On the other hand, using the invariance in space of a Lévy process, (1) and recalling that $r_n = 1/W_n(0)$, we get

$$\mathbf{P}_n^0\left(T(a) < T(0)\right) = \mathbf{P}_n^{-a}\left(T(0) < T(-a)\right) = \frac{1}{r_n W_n(-a)}.$$

The result therefore follows from Lemma 3.4.

Second case: $A = \{a, b\}$ with $a < 0 < b$. Under $\mathbf{P}_n^0$, the event $\{T(a) < T(b)\}$ is equal to the event that all excursions away from 0 before the first one that hits a (which exists since X does not drift to $+\infty$) do not hit b. Hence

$$\mathbf{P}_n^0\left(T(a) < T(b)\right) = \sum_{k \geq 0} \mathbf{P}_n^0\left(T(a) < T(0)\right) \left\{\mathbf{P}_n^0\left(T(0) < T(a) \wedge T(b)\right)\right\}^k$$

$$= \frac{\mathbf{P}_n^0\left(T(a) < T(0)\right)}{\mathbf{P}_n^0\left(T(a) \wedge T(b) < T(0)\right)}$$

so that

$$\mathbf{P}_n^0\left(T(a) \wedge T(b) < T(0)\right) = \frac{\mathbf{P}_n^0\left(T(a) < T(0)\right)}{\mathbf{P}_n^0\left(T(a) < T(b)\right)} = \frac{W_n(0)/W_n(-a)}{W_n(b)/W_n(b-a)} \quad (3)$$

using (1) for the last equality. Using the same reasoning, we derive a similar formula for $\mathcal{N}\left(T(a) \wedge T(b) < T(0)\right)$ as follows. Let η be the time of the first atom of the Poisson point process of excursions $(e_t, t \geq 0)$ of X away from 0 such that $\inf e_t < a$. Then, η is distributed like an exponential random variable with parameter $\mathcal{N}(\inf X < a)$, and the Poisson point process $(e_t, t < \eta)$ is independent of η and has intensity measure $\mathcal{N}(\cdot; \inf X > a)$. Thus, by a similar path decomposition as previously,

$$\mathbf{P}^0\left(T(a) < T(b)\right) = \mathbb{E}\left\{\exp\left(-\eta \mathcal{N}\left(\sup X > b, \inf X > a\right)\right)\right\}$$

$$= \frac{\mathcal{N}(\inf X < a)}{\mathcal{N}(\inf X < a) + \mathcal{N}(\sup X > b, \inf X > a)}$$

$$= \frac{\mathcal{N}(\inf X < a)}{\mathcal{N}\left(T(a) \wedge T(b) < T(0)\right)}$$

from which it follows that

$$\mathcal{N}\left(T(a) \wedge T(b) < T(0)\right) = \frac{\mathcal{N}(\inf X < a)}{\mathbf{P}^0\left(T(a) < T(b)\right)} = \frac{1/W(-a)}{W(b)/W(b-a)} \quad (4)$$

using $\mathcal{N}(\inf X < a) = 1/W(-a)$ which was proved in the first case. In view of (3) and (4), we get

$$\lim_{n \to +\infty} r_n \mathbf{P}_n^0\left(T(a) \wedge T(b) < T(0)\right) = \mathcal{N}\left(T(a) \wedge T(b) < T(0)\right).$$

This gives the result.

Third case: $A = \{a\}$ with $a > 0$. Remember that $X^0 = X(\,\cdot\,\wedge T(0))$, with $X^0 = X$ in the event $\{T(0) = +\infty\}$. Consider now $c < 0$: on the one hand, we have by definition

$$\mathbf{P}_n^0\left(T(c) \wedge T(a) < T(0)\right) = \mathbf{P}_n^0\left(\inf X^0 < c \text{ or } \sup X^0 > a\right).$$

On the other hand, (3) for the first equality and (1) for the second one give

$$\mathbf{P}_n^0\left(T(c) \wedge T(a) < T(0)\right) = \frac{W_n(0)/W_n(-c)}{W_n(a)/W_n(a-c)} = \frac{\mathbf{P}_n^a(T(0) < T(a-c))}{r_n W_n(a)}.$$

Because X under $\mathbf{P}_n^a$ does not drift to $+\infty$, $\mathbf{P}_n^a(T(0) < T(a-c)) \to 1$ as $c \to -\infty$. Thus, letting $c \to -\infty$, we obtain

$$\mathbf{P}_n^0\left(\inf X^0 = -\infty \text{ or } \sup X^0 > a\right) = \frac{1}{r_n W_n(a)}.$$

Since under $\mathbf{P}_n^0$, $\sup X^0 > 0$ implies $\inf X^0 > -\infty$, we have

$$\mathbf{P}_n^0\left(\inf X^0 = -\infty \text{ or } \sup X^0 > a\right) = \mathbf{P}_n^0\left(T(0) = +\infty\right) + \mathbf{P}_n^0\left(T(a) < T(0)\right).$$

Finally, one obtains $\mathbf{P}_n^0\left(T(0) = +\infty\right) = 1/(r_n W_n(\infty))$ by taking $a = 0$ and letting $b = +\infty$ in (1), and in particular

$$\mathbf{P}_n^0\left(T(a) < T(0)\right) = \frac{1}{r_n W_n(a)} - \frac{1}{r_n W_n(\infty)}.$$

Similar arguments also imply

$$\mathcal{N}\left(T(a) < T(0)\right) = \frac{1}{W(a)} - \frac{1}{W(\infty)}$$

and so the result follows from Lemma 3.4.   □

4 Convergence of the Finite-Dimensional Distributions

Proposition 4.1 establishes the convergence of the finite-dimensional distributions of the local time processes shifted at a positive level, from which we deduce the finite-dimensional convergence of the processes appearing in Theorems 2.3 and 2.4 in Corollaries 4.2 and 4.3.

Proposition 4.1 *Let $a_0 > 0$. Then, the two sequences of shifted processes $L^0(\,\cdot\, + a_0)$ under $\mathbf{P}_n^*(\,\cdot\mid T(a_0) < T(0))$ and $\mathbf{P}_n(\,\cdot\mid T(a_0) < T(0))$ converge in the sense of finite-dimensional distributions to $L^0(\,\cdot\, + a_0)$ under $\mathcal{N}(\,\cdot\mid T(a_0) < T(0))$ and $\underline{\mathcal{N}}(\,\cdot\mid T(a_0) < T(0))$, respectively.*

Proof Let A be a finite subset of $[a_0, \infty)$ with $a_0 = \min A$, $A_0 = A \cup \{0\}$ and let $(\mathbb{Q}_n, \mathbb{Q})$ be a pair of probability distributions either equal to

$$\left(\mathbf{P}_n^* (\cdot \mid T(a_0) < T(0)), \, \mathscr{N}(\cdot \mid T(a_0) < T(0)) \right)$$

or equal to

$$\left(\mathbf{P}_n(\cdot \mid T(a_0) < T(0)), \, \underline{\mathscr{N}}(\cdot \mid T(a_0) < T(0)) \right).$$

We show that $(L^0(a), a \in A)$ under $\mathbb{Q}_n$ converges weakly to $(L^0(a), a \in A)$ under $\mathbb{Q}$, which will prove the result.

Step 1. We begin by expressing the laws of $(L^0(a), a \in A)$ under $\mathbb{Q}_n$ and of $(L^0(a), a \in A)$ under $\mathbb{Q}$ in a convenient form, cf. (5) and (6) below. Let $M = (M_k, k \geq 1)$ be the sequence with values in A_0 which keeps track of the successively distinct elements of A visited by X before the first visit to 0. More specifically, let $\sigma_1 = T(A)$ and for $k \geq 1$ define recursively

$$\sigma_{k+1} = \begin{cases} \inf \left\{ t > \sigma_k : X(t) \in A_0 \backslash \{X(\sigma_k)\} \right\} & \text{if } X(\sigma_k) \in A, \\ \sigma_k & \text{else.} \end{cases}$$

Note that for every $k \geq 1$, σ_k under both $\mathbb{Q}_n$ and $\mathbb{Q}$ is almost surely finite, so the above definition makes sense (in this proof we will only work under $\mathbb{Q}_n$ or $\mathbb{Q}$). Let $M_k = X(\sigma_k)$ and for $a \in A$ define

$$S(a) = \sum_{k=1}^{\infty} \mathbb{1}_{\{M_k = a\}}.$$

For each $a \in A$, this sum is finite, since M under $\mathbb{Q}_n$ and $\mathbb{Q}$ only makes a finite number of visits to A before visiting 0. When $M_k \in A$, X accumulates some local time at M_k between times σ_k and σ_{k+1} before visiting $M_{k+1} \in A_0 \setminus \{M_k\}$. The amount of local time accumulated depends on whether we are working under $\mathbb{Q}_n$ or $\mathbb{Q}$.

Under $\mathbb{Q}_n$ and conditionally on $M_k \in A$, X reaches $M_k \in A$ at time σ_k and then returns to this point a geometric number of times before visiting $M_{k+1} \in A_0 \setminus \{M_k\}$. Identifying the parameter of the geometric random variables involved, one sees that $(L^0(a), a \in A)$ under $\mathbb{Q}_n$ can be written as follows (with the convention $\sum_1^0 = 0$):

$$(L^0(a), a \in A) = \left(\frac{1}{r_n} \sum_{k=1}^{S(a)} \left(1 + G_k^n(a) \right), a \in A \right) \tag{5}$$

where $G_k^n(a)$ is a geometric random variable with success probability $q_n(a)$ given by

$$q_n(a) = \mathbf{P}_n^a \left(T(A_0 \backslash \{a\}) < T(a) \right)$$

and the random variables $(G_k^n(a), k \geq 1, a \in A)$ are independent and independent of the vector $(S(a), a \in A)$.

Similarly, under $\mathbb{Q}$ and conditionally on $M_k = a \in A$, it is well known by excursion theory that X accumulates an amount of local time at level a between times σ_k and σ_{k+1} which is exponentially distributed with parameter $q(a)$ given by

$$q(a) = \mathcal{N}^a(T(A_0 \backslash \{a\}) < T(a)),$$

where $\mathcal{N}^a$ is the excursion measure of X away from a. Iterating this decomposition, one sees that $\left(L^0(a), a \in A\right)$ under $\mathbb{Q}$ can be written as follows:

$$(L^0(a), a \in A) = \left(\sum_{k=1}^{S(a)} E_k(a), a \in A \right) \tag{6}$$

where $E_k(a)$ is an exponential random variable with parameter $q(a)$ and the random variables $(E_k(a), k \geq 1, a \in A)$ are independent and independent of $(S(a), a \in A)$.

In view of the decompositions (5) and (6) and the independence of the random variables appearing in these sums, the result will be proved if we can show that each summand $r_n^{-1}(1 + G_k^n(a))$ converges in distribution to $E_k(a)$, and if we can show that the numbers of terms $(S(a), a \in A)$ under $\mathbb{Q}_n$ also converge to $(S(a), a \in A)$ under $\mathbb{Q}$.

Step 2. We prove that for each $a \in A$, $r_n^{-1}(1 + G_1^n(a))$ converges in distribution to $E_1(a)$. This is actually a direct consequence of Lemma 3.6 which implies that $r_n q_n(a) \to q(a)$ (using the invariance in space of Lévy processes). The following last step is devoted to proving the convergence of $(S(a), a \in A)$ under $\mathbb{Q}_n$ toward $(S(a), a \in A)$ under $\mathbb{Q}$.

Step 3. To show that $(S(a), a \in A)$ under $\mathbb{Q}_n$ converges toward $(S(a), a \in A)$ under $\mathbb{Q}$, it is sufficient to show that M under $\mathbb{Q}_n$ converges toward M under $\mathbb{Q}$. To prove this, we note that both under $\mathbb{Q}_n$ and $\mathbb{Q}$, M is a Markov chain living in the finite state space A_0 and absorbed at 0. Thus to show that M under $\mathbb{Q}_n$ converges to M under $\mathbb{Q}$, it is enough to show that the initial distributions and also the transition probabilities converge.

Let us prove the convergence of the initial distributions. We have $\mathbb{Q}_n(M_1 = 0) = \mathbb{Q}(M_1 = 0) = 0$ and for $a \in A$,

$$\mathbb{Q}_n(M_1 = a) = \mathbb{Q}_n\left(X(T(A_0)) = a\right) = \mathbb{Q}_n\left(X(T(A)) = a\right).$$

Similarly,

$$\mathbb{Q}(M_1 = a) = \mathbb{Q}\left(X(T(A_0)) = a\right) = \mathbb{Q}\left(X(T(A)) = a\right).$$

When $\mathbb{Q}_n = \mathbf{P}_n^*(\,\cdot\,|\,T(a_0) < T(0))$ and $\mathbb{Q} = \mathcal{N}(\,\cdot\,|\,T(a_0) < T(0))$, we have

$$\begin{aligned}
\mathbb{Q}_n\left(X(T(A)) = a\right) &= \mathbf{P}_n^*\left(X(T(A)) = a \mid T(a_0) < T(0)\right) \\
&= \mathbf{P}_n^0\left(X(T(A)) = a \mid T(A) < +\infty\right)
\end{aligned}$$

and so Lemma 3.2 gives

$$\lim_{n \to +\infty} \mathbb{Q}_n\left(X(T(A)) = a\right) = \mathbf{P}^0\left(X(T(A)) = a \mid T(A) < +\infty\right).$$

Since

$$\begin{aligned}
\mathbf{P}^0\left(X(T(A)) = a \mid T(A) < +\infty\right) &= \mathcal{N}\left(X(T(A)) = a \mid T(a_0) < T(0)\right) \\
&= \mathbb{Q}\left(X(T(A)) = a\right)
\end{aligned}$$

this proves the convergence of the initial distributions in this case. The second case $\mathbb{Q}_n = \mathbf{P}_n(\,\cdot\,|\,T(a_0) < T(0))$ and $\mathbb{Q} = \underline{\mathcal{N}}(\,\cdot\,|\,T(a_0) < T(0))$ follows similarly by considering the reflected process: We have then

$$\mathbb{Q}_n\left(X(T(A)) = a\right) = \mathbf{P}_n\left(X(T(A)) = a \mid T(a_0) < T(0)\right) = \underline{\mathbf{P}}_n^0\left(X(T(A)) = a\right)$$

and so Lemma 3.2 gives

$$\lim_{n \to +\infty} \mathbb{Q}_n\left(X(T(A)) = a\right) = \underline{\mathbf{P}}^0\left(X(T(A)) = a\right).$$

Since

$$\underline{\mathbf{P}}^0\left(X(T(A)) = a\right) = \underline{\mathcal{N}}\left(X(T(A)) = a \mid T(a_0) < T(0)\right) = \mathbb{Q}\left(X(T(A)) = a\right)$$

this proves the convergence of the initial distributions in this case as well.

It remains to show that transition probabilities also converge. Note that by definition, in contrast with the initial distributions, transition probabilities of M under $\mathbb{Q}_n$ and $\mathbb{Q}$ do not depend on the case considered. Since 0 is an absorbing state for M under $\mathbb{Q}_n$ and $\mathbb{Q}$, we only have to show that for any $a \in A$ and $b \in A_0$ with $a \neq b$, we have $\mathbb{Q}_n\left(M_{k+1} = b \mid M_k = a\right) \to \mathbb{Q}\left(M_{k+1} = b \mid M_k = a\right)$. On the one hand,

$$\mathbb{Q}_n\left(M_{k+1} = b \mid M_k = a\right) = \mathbf{P}_n^a\left(X(T(A_0 \setminus \{a\})) = b\right)$$

while on the other hand,

$$\mathbb{Q}\left(M_{k+1} = b \mid M_k = a\right) = \mathbf{P}^a\left(X(T(A_0 \setminus \{a\})) = b\right).$$

Note that $T(A_0 \setminus \{a\})$ is almost surely finite for $a \in A$ both under $\mathbf{P}_n^a$ and $\mathbf{P}^a$, so the result follows from Lemma 3.2.   □

Corollary 4.2 *For any $a_0 > 0$, the two sequences of processes L^0 under $\mathbf{P}_n^* (\cdot \mid T(a_0) < T(0))$ and $\mathbf{P}_n (\cdot \mid T(a_0) < T(0))$ converge in the sense of finite-dimensional distributions to L^0 under $\mathcal{N}(\cdot \mid T(a_0) < T(0))$ and $\underline{\mathcal{N}}(\cdot \mid T(a_0) < T(0))$, respectively.*

Proof Let $I \geq 1$, $0 < a_1 < \cdots < a_I$ and $u_i > 0$ for $i = 0, \ldots, I$: We prove the result for the convergence under $\mathbf{P}_n^*$, the result for $\mathbf{P}_n$ follows along the same lines, replacing $\mathbf{P}_n^*$ by $\mathbf{P}_n$ and $\mathcal{N}$ by $\underline{\mathcal{N}}$. We show that

$$\lim_{n \to +\infty} \mathbf{P}_n^* \left(L^0(a_i) \geq u_i, i = 0, \ldots, I \mid T(a_0) < T(0) \right)$$
$$= \mathcal{N} \left(L^0(a_i) \geq u_i, i = 0, \ldots, I \mid T(a_0) < T(0) \right).$$

If $a_0 \leq a_1$, then this follows directly from Proposition 4.1. If $a_0 > a_1$, we use Bayes formula:

$$\mathbf{P}_n^* \left(L^0(a_i) \geq u_i, i = 0, \ldots, I \mid T(a_0) < T(0) \right)$$
$$= \frac{\mathbf{P}_n^* (T(a_1) < T(0))}{\mathbf{P}_n^* (T(a_0) < T(0))} \mathbf{P}_n^* \left(L^0(a_i) \geq u_i, i = 0, \ldots, I \mid T(a_1) < T(0) \right).$$

Lemmas 2.1 and 3.6 (use Lemma 3.3 for $\mathbf{P}_n$) give

$$\lim_{n \to +\infty} \frac{\mathbf{P}_n^* (T(a_1) < T(0))}{\mathbf{P}_n^* (T(a_0) < T(0))} = \frac{\mathcal{N} (T(a_1) < T(0))}{\mathcal{N} (T(a_0) < T(0))}$$

and so Proposition 4.1 gives

$$\lim_{n \to +\infty} \mathbf{P}_n^* \left(L^0(a_i) \geq u_i, i = 0, \ldots, I \mid T(a_0) < T(0) \right)$$
$$= \frac{\mathcal{N} (T(a_1) < T(0))}{\mathcal{N} (T(a_0) < T(0))} \mathcal{N} \left(L^0(a_i) \geq u_i, i = 0, \ldots, I \mid T(a_1) < T(0) \right)$$

which is equal to $\mathcal{N}(L^0(a_i) \geq u_i, i = 0, \ldots, I \mid T(a_0) < T(0))$. The result is proved. $\square$

Corollary 4.3 *Let $\zeta > 0$ and (z_n) be any integer sequence such that $\zeta_n = z_n/r_n \to \zeta$, and recall that $T^L(\zeta) = \inf \{t \geq 0 : L(0, t) \geq \zeta\}$. Then the sequence of processes $L(\cdot, T^L(\zeta_n))$ under $\mathbf{P}_n^0(\cdot \mid T^L(\zeta_n) < +\infty)$ converges in the sense of finite-dimensional distributions to $L(\cdot, T^L(\zeta))$ under $\mathbf{P}^0(\cdot \mid T^L(\zeta) < +\infty)$.*

Proof Since the marginals at 0 are deterministic and converge to ζ, we restrict our attention to finite sets $A \subset (0, \infty)$ and we are interested in the weak convergence of the sequence $(L(a, T^L(\zeta_n)), a \in A)$ under $\mathbf{P}_n^0(\cdot \mid T^L(\zeta_n) < +\infty)$. Let $a_0 = \min A > 0$: Then, only those excursions reaching level a_0 contribute and so for each

$n \geq 1$, $(L(a, T^L(\zeta_n)), a \in A)$ under $\mathbf{P}_n^0(\,\cdot\, | \, T^L(\zeta_n) < +\infty)$ is equal in distribution to

$$\left(\sum_{k=1}^{K_n} L_{n,k}^{a_0}(a), a \in A \right)$$

where $(L_{n,k}^{a_0}, k \geq 1)$ are i.i.d. processes with distribution L^0 under $\mathbf{P}_n^*(\,\cdot\, | \, T(a_0) < T(0))$, and K_n is an independent random variable distributed as a binomial random variable with parameters z_n and $\mathbf{P}_n^0\,(T(a_0) < T(0))$. Since

$$\lim_{n \to +\infty} z_n/r_n = \zeta \quad \text{and} \quad \lim_{n \to +\infty} r_n \mathbf{P}_n^0(T(a_0) < T(0)) = \mathcal{N}(T(a_0) < T(0)),$$

the second convergence being given by Lemma 3.6, the sequence (K_n) converges weakly to a Poisson random variable K with parameter $\zeta \mathcal{N}(T(a_0) < T(0))$. On the other hand, $L_{n,1}^{a_0}$ converges in distribution to L^0 under $\mathcal{N}(\,\cdot\, | \, T(a_0) < T(0))$ by Proposition 4.1, so the sequence $(L(a, T^L(\zeta_n)), a \in A)$ under $\mathbf{P}_n^0(\,\cdot\, | \, T^L(\zeta_n) < +\infty)$ converges weakly to

$$\left(\sum_{k=1}^{K} L_k^{a_0}(a), a \in A \right)$$

where $(L_k^{a_0}, k \geq 1)$ are independent from K and are i.i.d. with common distribution L^0 under $\mathcal{N}(\,\cdot\, | \, T(a_0) < T(0))$. This proves the result. Indeed, under $\mathbf{P}^0(\,\cdot\, | \, T^L(\zeta) < +\infty)$ also, only excursions reaching level a_0 contribute to $L(\,\cdot\,, T^L(\zeta))$ on A, and there is a Poisson number, say K', with parameter $\zeta \mathcal{N}(T(a_0) < T(0))$, of such excursions. These excursions are i.i.d. with common law $\mathcal{N}(\,\cdot\, | \, T(a_0) < T(0))$, and so their local time processes are i.i.d. with common distribution L^0 under $\mathcal{N}(\,\cdot\, | \, T(a_0) < T(0))$. The local time being an additive functional, it is the sum of local times of these K' i.i.d. excursions.   □

5 Tightness Results

Tightness is a delicate issue. In the finite variance case and assuming that the limiting Brownian motion (with drift) drifts to $-\infty$, it follows quickly from a simple queueing argument, see [29]. In the infinite variance case of the present paper, we prove tightness under the tightness condition stated in Sect. 2. So, in the rest of this section, we assume that the tightness condition holds, i.e., for each $n \geq 1$ we have $s_n = n^{1/\alpha}$ and $\Lambda_n = \Lambda$ where Λ has tail distribution $\mathbb{P}(\Lambda \geq s) = (1 + s)^{-\alpha}$. The main technical result concerning tightness is contained in the following proposition, "Appendix" is devoted to its proof.

Proposition 5.1 *For any $a_0 > 0$, the sequence $L^0(a_0 + \cdot)$ under $\mathbf{P}_n^*(\,\cdot\, | \, T(a_0) < T(0))$ converges weakly to the continuous process $L^0(a_0 + \cdot)$ under $\mathcal{N}(\,\cdot\, | \, T(a_0) < T(0))$.*

We now prove the tightness of the sequences considered in Theorems 2.3 and 2.4. In the sequel, we say that a sequence (Z_n) under $\mathbb{Q}_n$ is C-tight if it is tight and any accumulation point is continuous. It is known, see for instance Chapter VI in [18], that this holds if and only if for any $m, \delta > 0$,

$$\lim_{a \to +\infty} \limsup_{n \to +\infty} \mathbb{Q}_n \left(\sup_{0 \le t \le m} |Z_n(t)| \ge a \right) = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \mathbb{Q}_n \left(\varpi_m(Z_n, \varepsilon) \ge \delta \right) = 0$$
(7)

where ϖ is the following modulus of continuity:

$$\varpi_m(f, \varepsilon) = \sup \{ |f(t) - f(s)| : 0 \le t, s \le m, |t - s| \le \varepsilon \}.$$

Lemma 5.2 *For each $n \ge 1$, consider on the same probability space an integer valued random variable K_n and a sequence of processes $(Z_{n,k}, k \ge 1)$. Assume that for each $k \ge 1$ the sequence $(Z_{n,k}, n \ge 1)$ is C-tight and that the sequence $(K_n, n \ge 1)$ is tight. Then, the sequence $(Z_{n,1} + \cdots + Z_{n,K_n}, n \ge 1)$ is C-tight.*

Proof Let $S_n = Z_{n,1} + \cdots + Z_{n,K_n}$: We must show that it satisfies (7). We show how to control the modulus of continuity, the supremum can be dealt with similar arguments. For any $0 \le s, t \le m$ with $|t - s| \le \varepsilon$, we have

$$|S_n(t) - S_n(s)| \le \sum_{k=1}^{K_n} \left| Z_{n,k}(t) - Z_{n,k}(s) \right| \le \sum_{k=1}^{K_n} \varpi_m(Z_{n,k}, \varepsilon)$$

and so we obtain the following bound, valid for any $\beta > 0$:

$$\mathbb{Q}_n \left(\varpi_m(S_n, \varepsilon) \ge \delta \right) \le \mathbb{Q}_n \left(\sum_{k=1}^{\beta} \varpi_m(Z_{n,k}, \varepsilon) \ge \delta \right) + \mathbb{Q}_n(K_n \ge \beta)$$

$$\le \sum_{k=1}^{\beta} \mathbb{Q}_n \left(\varpi_m(Z_{n,k}, \varepsilon) \ge \delta/\beta \right) + \mathbb{Q}_n(K_n \ge \beta).$$

Since each sequence $(Z_{n,k}, n \ge 1)$ is C-tight, this gives

$$\lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \mathbb{Q}_n \left(\varpi_m(S_n, \varepsilon) \ge \delta \right) \le \limsup_{n \to +\infty} \mathbb{Q}_n(K_n \ge \beta)$$

which goes to 0 as β goes to infinity since (K_n) is tight. The result is proved. □

For $n \ge 1$ and $a > 0$, we denote by $G_n(a)$ a geometric random variable with parameter $1 - W_n(0)/W_n(a)$ (remember that $W_n(0) = 1/r_n$), so that according to Lemma 3.4, the sequence $(W_n(0)G_n(a))$ converges in distribution to an exponential random variable with parameter $1/W(a)$. The following proposition, combined with Corollary 4.3, proves Theorem 2.4.

Proposition 5.3 *Let $\zeta > 0$ and (z_n) be any integer sequence such that $\zeta_n = z_n/r_n \to \zeta$. Then, the sequence of processes $L(\,\cdot\,, T^L(\zeta_n))$ under $\mathbf{P}_n^0(\,\cdot\mid T^L(\zeta_n) < +\infty)$ is C-tight.*

Proof Fix any $a_0 > 0$ and let τ_n be the time of the first visit to 0 after time $T(a_0, z_n)$, i.e., $\tau_n = \inf\{t \geq T(a_0, z_n) : X(t) = 0\}$, and K_n be the number of excursions of X away from 0 that reach level a_0 before time τ_n. Unless otherwise stated, we work implicitly under $\mathbf{P}_n^0(\,\cdot\mid T(a_0, z_n) < +\infty)$ so that τ_n and K_n are well defined. Then, the following decomposition holds (using Lemma 2.1):

$$L(a_0 + a, \tau_n) = \sum_{k=1}^{K_n} L_{n,k}^{a_0}(a), \quad a \geq 0, \tag{8}$$

where $(L_{n,k}^{a_0}, k \geq 1)$ are independent processes, all equal in distribution to $L^0(a_0 + \,\cdot\,)$ under $\mathbf{P}_n^*(\,\cdot\mid T(a_0) < T(0))$. Note that by Proposition 5.1, $(L_{n,1}^{a_0}, n \geq 1)$ is C-tight. In an excursion that reaches level a_0, the law of the number of visits of X to a_0 is $1 + G_n(a_0)$; thus, K_n is equal in distribution to

$$\min\left\{k \geq 1 : \sum_{i=1}^{k}\left(1 + G_{n,i}(a_0)\right) \geq \zeta_n\right\}$$

with $(G_{n,i}(a_0), i \geq 1)$ i.i.d. random variables with common distribution $G_n(a_0)$. Since $(G_n(a_0)/r_n)$ converges in distribution to an exponential random variable with parameter $1/W(a_0)$ and $z_n/r_n \to \zeta$, the sequence (K_n) converges in distribution to a Poisson random variable with parameter $\zeta/W(a_0)$. In particular, the sequence (K_n) is tight, and combining (8), Lemma 5.2, Proposition 5.1 and the C-tightness of $(L_{n,1}^{a_0}, n \geq 1)$, one sees that the sequence $L(a_0 + \,\cdot\,, \tau_n)$ is C-tight. We now prove the desired C-tightness of the sequence $L(\,\cdot\,, T^L(\zeta_n))$ under $\mathbf{P}_n^0(\,\cdot\mid T^L(\zeta_n) < +\infty)$.

For $a \geq 0$ define $L^{1,a_0}(a)$ and $L^{2,a_0}(a)$, the local time accumulated at level $a_0 + a$ in the time interval $[T(a_0), T(a_0, z_n)]$, and $[0, T(a_0)) \cup (T(a_0, z_n), \tau_n]$, respectively. Then, $L(a_0 + a, \tau_n) = L^{1,a_0}(a) + L^{2,a_0}(a)$ which we write as

$$L^{1,a_0}(a) = L(a_0 + a, \tau_n) - L^{2,a_0}(a), \ a \geq 0.$$

We have just proved that the sequence $L(a_0 + \,\cdot\,, \tau_n)$ was C-tight, so if we can prove that the sequence L^{2,a_0} is also C-tight, Lemma 5.2 will imply the C-tightness of L^{1,a_0}. But by invariance in space, L^{1,a_0} under $\mathbf{P}_n^0(\,\cdot\mid T(a_0, z_n) < +\infty)$ is equal in distribution to $L(\,\cdot\,, T^L(\zeta_n))$ under $\mathbf{P}_n^0(\,\cdot\mid T^L(\zeta_n) < +\infty)$, so this will prove the desired result. Hence, it remains to prove the C-tightness of L^{2,a_0}.

Let $g(a_0)$ be the left endpoint of the first excursion of X away from 0 that reaches a_0 and define the excursion Y as follows:

$$Y(t) = \begin{cases} X(t + g(a_0)) & \text{if } 0 \leq t \leq T(a_0) - g(a_0), \\ X\left((t + T(a_0, z_n) + g(a_0) - T(a_0)) \wedge \tau_n\right) & \text{if } t \geq T(a_0) - g(a_0). \end{cases}$$

Then, Y is distributed like $X(\cdot \wedge T(0))$ under $\mathbf{P}_n^*(\cdot \mid T(a_0) < T(0))$ and

$$L^{3,a_0}(a) \stackrel{\text{def.}}{=} L^{2,a_0}(a) + \frac{1}{r_n}\mathbb{1}_{\{a=a_0\}},$$

so that L^{3,a_0} is the local time process above level a_0 of the process Y. The term $\mathbb{1}_{\{a=a_0\}}$ is here to compensate the fact that $L^{2,a_0}(a)$ is the local time accumulated during the time interval $[0, T(a_0)) \cup (T(a_0, z_n), \tau_n)$ instead of $[0, T(a_0)) \cup [T(a_0, z_n), \tau_n)$, so L^{2,a_0} is missing one visit of Y to a_0. Thus, L^{3,a_0} is equal in distribution to the process $L^0(a_0 + \cdot)$ under $\mathbf{P}_n^*(\cdot \mid T(a_0) < T(0))$, and the sequence L^{3,a_0} is therefore C-tight by Proposition 5.1. Neglecting the factor r_n^{-1} which vanishes in the limit, this implies that L^{2,a_0} is C-tight and concludes the proof. □

The following proposition, combined with Corollary 4.3, proves Theorem 2.3.

Proposition 5.4 *For any $a_0 > 0$, the two sequences L^0 under $\mathbf{P}_n^*(\cdot \mid T(a_0) < T(0))$ and $\mathbf{P}_n(\cdot \mid T(a_0) < T(0))$ are C-tight.*

Proof In the rest of the proof fix some $a_0 > 0$. We first show the C-tightness of L^0 under $\mathbf{P}_n^*(\cdot \mid T(a_0) < T(0))$, from which the C-tightness under $\mathbf{P}_n(\cdot \mid T(a_0) < T(0))$ is then derived. Let $z_n = \lceil r_n \rceil$ be the smallest integer larger than r_n and $\zeta_n = z_n/r_n$, so that $\zeta_n \to 1$. Since $T^L(\zeta_n) = T(0, z_n)$, when $T^L(\zeta_n)$ is finite X has at least z_n excursions away from 0, and so we can define K_n the number of excursions among these z_n first excursions that visit a_0. Let $L_n = L(\cdot, T^L(\zeta_n))$ and $\mathbb{Q}_n = \mathbf{P}_n^0(\cdot \mid T^L(\zeta_n) < +\infty, K_n \geq 1)$: using Proposition 5.3, it is easy to show that L_n under $\mathbb{Q}_n$ is C-tight. Indeed, decomposing the path $(X(t), 0 \leq t \leq T(0, z_n))$ into its z_n excursions away from 0, one gets

$$\mathbf{P}_n^0\left(K_n = 0 \mid T^L(\zeta_n) < +\infty\right) = \left\{\mathbf{P}_n^0\left(T(0) < T(a_0) \mid T(0) < +\infty\right)\right\}^{z_n}.$$

By duality and (1),

$$\mathbf{P}_n^0\left(T(0) < T(a_0) \mid T(0) < +\infty\right) = \mathbf{P}_n^0\left(T(0) < T(-a_0) \mid T(0) < +\infty\right)$$
$$= \frac{1 - \frac{1}{r_n W_n(a_0)}}{1 - \frac{1}{r_n W_n(\infty)}}$$

which gives $\mathbf{P}_n^0\left(K_n = 0 \mid T^L(\zeta_n) < +\infty\right) \to e^{-1/W(a_0)}e^{1/W(\infty)} < 1$. It follows in particular that $C = 1/\left(\inf_n \mathbf{P}_n^0\left(K_n \geq 1 \mid T^L(\zeta_n) < +\infty\right)\right)$ is finite, and so for any $m, \varepsilon, \delta > 0$,

$$\mathbb{Q}_n\left(\varpi_m(L_n, \varepsilon) \geq \delta\right) = \frac{\mathbf{P}_n^0\left(\varpi_m(L_n, \varepsilon) \geq \delta, K_n \geq 1 \mid T^L(\zeta_n) < +\infty\right)}{\mathbf{P}_n^0\left(K_n \geq 1 \mid T^L(\zeta_n) < +\infty\right)}$$
$$\leq C\mathbf{P}_n^0\left(\varpi_m(L_n, \varepsilon) \geq \delta \mid T^L(\zeta_n) < +\infty\right).$$

Since L_n under $\mathbf{P}_n^0\left(\cdot \mid T^L(\zeta_n) < +\infty\right)$ is C-tight by Proposition 5.3, this implies that the second condition in (7) holds. One can similarly control the supremum and prove that the first condition in (7) also holds, which finally proves the C-tightness of L_n under $\mathbb{Q}_n$.

We now prove the C-tightness of L^0 under $\mathbf{P}_n^*(\cdot \mid T(a_0) < T(0))$. Let $g(a_0) < d(a_0)$ be the endpoints of the first excursion of X away from 0 which reaches level a_0. In the event $\{T^L(\zeta_n) < +\infty, K_n \geq 1\}$, define L^1 as the local time process in the interval $[0, g(a_0)) \cup (d(a_0), T(0, z_n)]$ and L^2 the local time process in the interval $[g(a_0), d(a_0)]$. Then under $\mathbb{Q}_n$, the three following properties hold:

(i) the process $(L^1(a) + r_n^{-1}\mathbb{1}_{\{a=0\}}, a \geq 0)$ is equal in distribution to $L(\cdot, T^L(\zeta_n'))$ under $\mathbf{P}_n^0(\cdot \mid T^L(\zeta_n') < +\infty)$, where $\zeta_n' = (z_n - 1)/r_n$;
(ii) $L^2 = L_n - L^1$;
(iii) L^2 is equal in distribution to L^0 under $\mathbf{P}_n^*(\cdot \mid T(a_0) < T(0))$.

Proposition 5.3 and the first property entail that L^1 under $\mathbb{Q}_n$ is C-tight. Since L_n under $\mathbb{Q}_n$ has been proved to be C-tight, the second property together with Lemma 5.2 implies that L^2 under $\mathbb{Q}_n$ is C-tight. The last property finally proves the C-tightness of L^0 under $\mathbf{P}_n^*(\cdot \mid T(a_0) < T(0))$.

We now prove that L^0 under $\mathbf{P}_n(\cdot \mid T(a_0) < T(0))$ is C-tight using the Radon–Nikodym derivative. Let $f(s) = \mathbb{P}(\Lambda \geq s)$: then $\mathbb{P}(\Lambda^* \in \mathrm{d}s)/\mathrm{d}s = f(s)$ and $\mathbb{P}(\Lambda \in \mathrm{d}s)/\mathrm{d}s = -f'(s)$ so that Λ is absolutely continuous with respect to Λ^* with Radon–Nikodym derivative $-f'/f$. In particular, for any measurable function g, it holds that

$$\mathbb{E}\left(g(\Lambda)\right) = \mathbb{E}\left(-\frac{f'(\Lambda^*)}{f(\Lambda^*)}g(\Lambda^*)\right).$$

Consequently, since $-f'(s)/f(s) = \alpha/(1 + s)$, we get for any m, ε and $\delta > 0$

$$\mathbf{P}_n\left(\varpi_m(L^0, \varepsilon) \geq \delta \mid T(a) < T(0)\right) = \alpha\mathbb{E}\left(\frac{g(\Lambda^*)}{1 + \Lambda^*}\right) \leq \alpha\mathbf{P}_n^*\left(\varpi_m(L^0, \varepsilon) \geq \delta \mid T(a) < T(0)\right)$$

where $g(x) = \mathbf{P}_n^x\left(\varpi_m(L^0, \varepsilon) \geq \delta \mid T(a) < T(0)\right)$. Since L^0 under $\mathbf{P}_n^*(\cdot \mid T(a_0) < T(0))$ is C-tight by the first part of the proof, we deduce that the second condition in (7) is satisfied. The supremum can be handled similarly, and therefore, the proof is complete. $\square$

6 Implications for Branching Processes and Queueing Theory

6.1 Implications for Branching Processes

A Crump–Mode–Jagers (CMJ) process, or general branching process, is a stochastic process with non-negative integer values counting the size of a population where individuals give birth to independent copies of themselves; see for instance [16] for a definition.

For $n \geq 1$ and $z \in \mathbb{N}$, let $\mathbb{Z}_n^z$ (resp. $\mathbb{Z}_n^{z*}$) be the law of a binary, homogeneous CMJ branching process with life length distribution Λ_n and offspring intensity κ_n started with z individuals with i.i.d. life lengths with common distribution Λ_n (resp. Λ_n^*). Let $\mathbf{Z}_n^z$ (resp. $\mathbf{Z}_n^{z*}$) be the law of $X(s_n t)/r_n$ under $\mathbb{Z}_n^z$ (resp. $\mathbb{Z}_n^{z*}$). It follows directly from results in [29] that $\mathbf{Z}_n^1$ is the law of L^0 under $\mathbf{P}_n$ and that $\mathbf{Z}_n^{z*}$ is the law of $L(\cdot, T^L(z/r_n))$ under $\mathbf{P}_n^0(\cdot \mid T^L(z/r_n) < +\infty)$. In the sequel, for $\zeta > 0$ introduce $\mathbf{Z}^{\zeta*}$ the law of $L(\cdot, T^L(\zeta))$ under $\mathbf{P}^0(\cdot \mid T^L(\zeta) < +\infty)$ and $\mathbf{Z}^*$ (resp. $\mathbf{Z}$) the push-forward of $\mathscr{N}$ (resp. $\underline{\mathscr{N}}$) by L^0. In other words, $\mathbf{Z}^*$ and $\mathbf{Z}$ are defined by

$$\mathbf{Z}^*(A) = \mathscr{N}(L^0 \in A) \quad \text{and} \quad \mathbf{Z}(A) = \underline{\mathscr{N}}(L^0 \in A)$$

for any Borel set $A \subset \mathscr{E}$. The two theorems below are therefore plain reformulations of Theorem 2.3 and 2.4.

Theorem 6.1 *For any $\varepsilon > 0$, the two sequences $\mathbf{Z}_n^{1*}(\cdot \mid T(0) > \varepsilon)$ and $\mathbf{Z}_n^1(\cdot \mid T(0) > \varepsilon)$ converge in the sense of finite-dimensional distributions to $\mathbf{Z}^*(\cdot \mid T(0) > \varepsilon)$ and $\mathbf{Z}(\cdot \mid T(0) > \varepsilon)$, respectively. If in addition, the tightness assumption holds, then both convergences hold in the sense of weak convergence.*

Theorem 6.2 *Let $\zeta > 0$ and (z_n) be any integer sequence such that $z_n/r_n \to \zeta$. Then, the sequence $\mathbf{Z}_n^{z_n*}$ converges in the sense of finite-dimensional distributions to $\mathbf{Z}^{\zeta*}$. If in addition, the tightness assumption holds, then the convergence holds in the sense of weak convergence.*

Since there is a rich literature on the scaling limits of branching processes, it is interesting to put this result in perspective. CMJ branching processes are (possibly non-Markovian) generalizations of Galton–Watson (GW) branching processes in continuous time. Scaling limits of GW processes have been exhaustively studied since the pioneering work of Lamperti [31], see [15]. Accumulation points of sequences of renormalized GW processes are called CSBP, they consist of all the continuous time, continuous state space, time-homogeneous Markov processes which satisfy the branching property. Via the Lamperti transformation, they are in one-to-one correspondence with spectrally positive Lévy processes killed upon reaching 0, see [30] or [8].

On the other hand, little is known about scaling limits of CMJ branching processes, except for the Markovian setting where individuals live for an exponential duration and give birth, upon death, to a random number of offspring. Intuitively, in this case, the tree representing the CMJ process should not differ significantly from the corresponding genealogical GW tree because the life length distribution has a light tail. And indeed, correctly renormalized, Markovian CMJ processes converge to CSBP, see [17]. The same intuition explains results obtained by Sagitov [37,38], who proves the convergence of the finite-dimensional distributions of some non-Markovian CMJ processes toward CSBP. It also provides an explanation for the results obtained by Lambert et al. [29], where it is proved that binary, homogeneous CMJ branching processes whose life length distribution has a finite variance converge to Feller diffusion, the only CSBP with continuous sample paths.

In the infinite variance case studied in the present paper, and with which Theorems 6.1 and 6.2 are concerned, some individuals will intuitively live for a very long time, causing the tree representing the CMJ to differ significantly from the corresponding genealogical GW tree, a difference that should persist in the limit. Our results are consistent with this intuition, since the CMJ studied here converge to non-Markovian processes (see remark following Theorem 2.4). To the best of our knowledge, this is the first time that a sequence of branching processes converging to a non-Markovian limit has been studied.

6.2 Implications for Queueing Theory

The Processor-Sharing queue is the single-server queue in which the server splits its service capacity equally among all the users present. For $n \geq 1$ and $z \in \mathbb{N}$, let $\mathbb{Q}_n^z$ (resp. $\mathbb{Q}_n^{z*}$) be the law of the queue length process of the Processor-Sharing queue with Poisson arrivals at rate κ_n, service distribution Λ and started with z initial customers with i.i.d. initial service requirements with common distribution Λ (resp. Λ^*). Let $\mathbf{Q}_n^z$ (resp. $\mathbf{Q}_n^{z*}$) be the law of $X(nt)/n^{1-1/\alpha}$ under $\mathbb{Q}_n^z$ (resp. $\mathbb{Q}_n^{z*}$).

Let $\mathscr{E}_+ \subset \mathscr{E}$ be the set of positive excursions with finite length. Let $\mathscr{L} : \mathscr{E}_+ \to \mathscr{E}_+$ be the Lamperti transformation: by definition $\mathscr{L}(e)$ for $e \in \mathscr{E}_+$ is the only positive excursion that satisfies $\mathscr{L}(e)(\int_0^t e) = e(t)$ for every $t \geq 0$. In the rest of this section, if μ is some positive measure on $\mathscr{E}_+$, write $\mathscr{L}(\mu)$ for the push-forward of μ by $\mathscr{L}$: for any Borel set A,

$$\mathscr{L}(\mu)(A) = \mu(\mathscr{L} \in A).$$

Recall that $X^0 = X(\cdot \wedge T(0))$: We have the following result, see for instance Chapter 7.3 in [36] and references therein.

Lemma 6.3 *For any integer $z \geq 0$ and any $n \geq 1$, X^0 under $\mathbf{Q}_n^z$ (resp. $\mathbf{Q}_n^{z*}$) is equal in distribution to $\mathscr{L}(\mathbf{Z}_n^z)$ (resp. $\mathscr{L}(\mathbf{Z}_n^{z*})$).*

In view of Theorems 6.1, 6.2 and Lemma 6.3, it is natural to expect excursions of the queue length process to converge, upon suitable conditioning. The conditioning $\{T(0) > \varepsilon\}$ of $\mathbf{Z}_n^1$ as in Theorem 6.1 is however not convenient in combination with the map $\mathscr{L}$. Instead, it will be more convenient to consider $\mathbf{Z}_n^1(\cdot \mid T \circ \mathscr{L} > \varepsilon)$, where from now on we sometimes write $T = T(0)$, so that $T \circ \mathscr{L}(e) = T_{\mathscr{L}(e)}(0)$ for $e \in \mathscr{E}_+$.

From the definition of $\mathscr{L}$, one can see that, for $e \in \mathscr{E}_+$, we have $\mathscr{L}(e)(t) = 0$ if and only if $t \geq \int_0^{T_e(0)} e$, i.e., $T \circ \mathscr{L}(e) = \int_0^{T_e(0)} e = \int_0^\infty e$. In particular, when L^0 is well defined, we have

$$T \circ \mathscr{L} \circ L^0 = \int_0^\infty L^0 = \int_0^\infty L(a, T) \mathrm{d}a = T. \tag{9}$$

Lemma 6.4 *For any $\varepsilon > 0$, we have $\mathbf{Z}_n^1(\cdot \mid T \circ \mathscr{L} > \varepsilon) \Rightarrow \mathbf{Z}(\cdot \mid T \circ \mathscr{L} > \varepsilon)$.*

Proof Starting from Theorem 6.1 and using Lemma 4.7 in [28], one sees that it is enough to show that for any $\varepsilon > 0$,

$$\lim_{n \to +\infty} \kappa_n s_n \mathbf{Z}_n^1 (T > \varepsilon) = \mathbf{Z}(T > \varepsilon), \quad \lim_{n \to +\infty} \kappa_n s_n \mathbf{Z}_n^1 (T \circ \mathscr{L} > \varepsilon) = \mathbf{Z}(T \circ \mathscr{L} > \varepsilon)$$
(10)

and

$(X, T \circ \mathscr{L})$ under $\mathbf{Z}_n^1(\,\cdot\, | \, T > \varepsilon)$ converges weakly to $(X, T \circ \mathscr{L})$ under $\mathbf{Z}(\,\cdot\, | \, T > \varepsilon)$.
(11)

Since $\mathbf{Z}_n^1(T > \varepsilon) = \mathbf{P}_n(T(\varepsilon) < T(0))$ and $\mathbf{Z}(T > \varepsilon) = \underline{\mathscr{N}}(T(\varepsilon) < T(0))$, the first limit in (10) is given by Lemma 3.3. Similarly, (9) entails $\mathbf{Z}_n^1(T \circ \mathscr{L} > \varepsilon) = \mathbf{P}_n(T > \varepsilon)$ and $\mathbf{Z}(T \circ \mathscr{L} > \varepsilon) = \underline{\mathscr{N}}(T > \varepsilon)$ and so the second limit in (10) follows from Lemma 3.5. Thus, to complete the proof, it remains to prove (11).

Since T under $\mathbf{Z}_n^1(\,\cdot\, | \, T > \varepsilon)$ is equal in distribution to $\sup X^0$ under $\mathbf{P}_n(\,\cdot\, | \, \sup X^0 > \varepsilon)$, one can show that it converges weakly to T under $\mathbf{Z}(\,\cdot\, | \, T > \varepsilon)$. Using similar arguments as in the proof of Lemma 3.2, one can strengthen this to get the joint convergence of (X, T). Since $T(\mathscr{L}(e)) = \int_0^{T_e(0)} e$ and the map $(f, t) \in \mathscr{D} \times [0, \infty) \mapsto \int_0^t f$ is continuous, the continuous mapping theorem implies that $(X, T \circ \mathscr{L})$ under $\mathbf{Z}_n^1(\,\cdot\, | \, T > \varepsilon)$ converges weakly to $(X, T \circ \mathscr{L})$ under $\mathbf{Z}(\,\cdot\, | \, T > \varepsilon)$. ☐

We now state the two main queueing results of the paper. The first one (Theorem 6.5) is a direct consequence of Lemma 6.4 together with continuity properties of the map $\mathscr{L}$, it gives results on excursions of the Processor-Sharing queue length process. The second one (Theorem 6.6) leverages on results in [28] to give two simple conditions under which not only excursions but the full processes converge.

Theorem 6.5 *Let $\varepsilon, \zeta > 0$ and (z_n) be any integer sequence such that $z_n/r_n \to \zeta$. Then, the two sequences of processes X^0 under $\mathbf{Q}_n^{z_n*}$ and $\mathbf{Q}_n^1(\,\cdot\, | \, T > \varepsilon)$ converge weakly to $\mathscr{L}(\mathbf{Z}^{\zeta*})$ and $\mathscr{L}(\mathbf{Z})(\,\cdot\, | \, T > \varepsilon)$, respectively.*

Proof Lemma 6.3 shows that X^0 under $\mathbf{Q}_n^{z_n*}$ and $\mathbf{Q}_n^1(\,\cdot\, | \, T > \varepsilon)$ is equal in distribution to $\mathscr{L}(\mathbf{Z}_n^{z_n*})$ and $\mathscr{L}(\mathbf{Z}_n^1(\,\cdot\, | \, T \circ \mathscr{L} > \varepsilon))$, respectively. Then, note that since we consider the Processor-Sharing queue with Poisson arrivals at rate κ_n and service distribution Λ, the tightness assumption holds. Thus, $\mathbf{Z}_n^1(\,\cdot\, | \, T \circ \mathscr{L} > \varepsilon) \Rightarrow \mathbf{Z}(\,\cdot\, | \, T \circ \mathscr{L} > \varepsilon)$ by Lemma 6.4 and $\mathbf{Z}_n^{z_n*} \Rightarrow \mathbf{Z}^{\zeta*}$ by Theorem 6.2. Hence, the result would be proved if we could show that $\mathscr{L}$ was continuous along the two sequences $\mathbf{Z}_n^1(\,\cdot\, | \, T \circ \mathscr{L} > \varepsilon)$ and $\mathbf{Z}_n^{z_n*}$ (using that $\mathscr{L}(\mathbf{Z}(\,\cdot\, | \, T \circ \mathscr{L} > \varepsilon)) = \mathscr{L}(\mathbf{Z})(\,\cdot\, | \, T > \varepsilon)$). Lemma 2.5 in [29] shows that this holds if the two sequences T under $\mathbf{Z}_n^{z_n*}$ and $\mathbf{Z}_n^1(\,\cdot\, | \, T \circ \mathscr{L} > \varepsilon)$ are tight; actually, one can show as easily that they converge weakly.

Indeed, defining $\zeta_n = z_n/r_n$, one sees that T under $\mathbf{Z}_n^{z_n*}$ is equal in distribution to $\sup_{[0, T^L(\zeta_n)]} X$ under $\mathbf{P}_n^0(\,\cdot\, | \, T^L(\zeta_n) < +\infty)$. One can show using standard arguments that this converges to $\sup_{[0, T^L(\zeta)]} X$ under $\mathbf{P}^0(\,\cdot\, | \, T^L(\zeta) < +\infty)$, which is equal in distribution to T under $\mathbf{Z}^{\zeta*}$ and thus shows the weak convergence of T under $\mathbf{Z}_n^{z_n*}$.

Similar arguments for T under $\mathbf{Z}_n^1(\,\cdot\mid T\circ\mathscr{L}>\varepsilon)$ give the result, using also (9) in this case.   □

Theorem 6.6 *If one of the following two conditions is met:*

- *either the sequence $\mathbf{Q}_n^0$ is tight;*
- *or for any $\eta>0$,*

$$\lim_{\varepsilon\to 0}\limsup_{n\to+\infty}\left[s_n\mathbf{Q}_n^1\left(\sup X^0\geq\eta,\,T\leq\varepsilon\right)\right]=0;\qquad(12)$$

then for any $\zeta\geq 0$ and any integer sequence (z_n) such that $z_n/r_n\to\zeta$, the sequence $\mathbf{Q}_n^{z_n*}$ converges weakly to the unique regenerative process that starts at ζ and such that its excursion measure is $\mathscr{L}(\mathbf{Z})$, its zero set has zero Lebesgue measure and, when $\zeta>0$, its first excursion is equal in distribution to $\mathscr{L}(\mathbf{Z}^{\zeta*})$.

Proof We check that all the assumptions of Theorem 3 in [28] are satisfied. In order to help the reader, we provide a translation of the notation used in [28]: the notation $\mathscr{N}$, φ, c_n, $\mathscr{N}_n$, b_n, and v_∞ appearing in the statement of [28, Theorem 3] correspond, respectively, to (with the notation of the present paper) $\mathscr{L}(\mathbf{Z})$, T, $s_n\kappa_n$, $\mathbf{Q}_n^1$, $n\kappa_n$ and sup.

That $\mathscr{L}(\mathbf{Z})(T=+\infty)=\mathbf{Q}_n^1(T=+\infty)=0$ follows from the fact that X under $\mathbf{P}^0$ does not drift to $+\infty$. Let $\varepsilon>0$: The assumption (**H1**) in [28] corresponds to $s_n\kappa_n\mathbf{Q}_n^1(T>\varepsilon)\to\mathscr{L}(\mathbf{Z})(T>\varepsilon)$. Since the workload process associated with X under $\mathbf{Q}_n^1$ and renormalized in space by s_n is equal in distribution to $\underline{X}$ under $\mathbf{P}_n$, we have $\mathbf{Q}_n^1(T>\varepsilon)=\mathbf{P}_n(T>\varepsilon)$. Moreover, (9) gives $\mathscr{L}(\mathbf{Z})(T>\varepsilon)=\underline{\mathscr{N}}(T>\varepsilon)$ and so $s_n\kappa_n\mathbf{Q}_n^1(T>\varepsilon)\to\mathscr{L}(\mathbf{Z})(T>\varepsilon)$ is just a restatement of Lemma 3.5. It can be shown similarly that the assumption (**H2**) in [28] holds (i.e., for any $\varepsilon,\lambda>0$ we have $s_n\kappa_n\mathbf{Q}_n^1(1-e^{-\lambda T};T\leq\varepsilon)\to\mathscr{L}(\mathbf{Z})(1-e^{-\lambda T};T\leq\varepsilon)$).

The convergence of X^0 under $\mathbf{Q}_n^1(\,\cdot\mid T>\varepsilon)$ (which corresponds to $e_\varepsilon(X_n)$ in [28]) has been taken care of by Theorem 6.5. It is easily deduced that (X^0,T) under $\mathbf{Q}_n^1(\,\cdot\mid T>\varepsilon)$ converges toward (X^0,T) under $\mathscr{L}(\mathbf{Z})(\,\cdot\mid T>\varepsilon)$ (which corresponds to $(e_\varepsilon,\varphi\circ e_\varepsilon)(X_n)\Rightarrow(e_\varepsilon,\varphi\circ e_\varepsilon)(X)$ in [28]).

Finally, our assumption (12) corresponds exactly to the last condition (6) in [28], which proves the result if (12) is assumed. If one assumes tightness of $\mathbf{Q}_n^0$ instead of (12), one can check that the proof of Theorem 3 in [28] goes through, since the assumption (6) there is only used to show tightness of $\mathbf{Q}_n^0$. The proof is therefore complete.   □

6.3 A Relation with the Height Process

Let Q be the regenerative process, started at 0, whose zero set has zero Lebesgue measure and with excursion measure $\mathscr{L}(\mathbf{Z})$. Duquesne and Le Gall [12] have introduced a process H, which they call the height process, that codes the genealogy of a CSBP. It is interesting to note that for every $t\geq 0$, $Q(t)$ and $H(t)$ are equal in distribution, as can be seen by combining results from Kella et al. [21] and Limic [32]. Indeed, the first result shows that the one-dimensional distributions of a Processor-Sharing queue and

a Last-In-First-Out (LIFO) queue started empty are equal. As for the second result, it shows that in the finite variance case a LIFO queue suitably renormalized converges to H (which is then just a reflected Brownian motion), and the author argues in the remark preceding Theorem 5 that this result can be extended to the α-stable setting which we have also studied here.

It would be interesting to study whether Q and H share more similarities. In general, however, these processes may be dramatically different. For instance, it follows from Lemma 2.2 and Duquesne and Le Gall [12] that if Ψ is such that

$$\int^\infty \frac{d\lambda}{\Psi(\lambda)\sqrt{\log\lambda}} < +\infty \quad \text{while} \quad \int^\infty \frac{d\lambda}{\Psi(\lambda)} = +\infty$$

then H is very wild (not even càdlàg) while Q is continuous.

7 Appendix: Proof of Proposition 5.1

In the rest of this section, we fix some $a_0 > 0$ and we assume that the tightness assumption stated in Sect. 2 holds: In particular, $\Lambda_n = \Lambda$ with $\mathbb{P}(\Lambda \geq s) = (1+s)^{-\alpha}$ for some $1 < \alpha < 2$, $n = s_n^\alpha$ and $r_n = s_n^{\alpha-1}$. The goal of this section is to prove that the sequence $L^0(a_0 + \cdot)$ under $\mathbf{P}_n^*(\cdot \mid T(a_0) < T(0))$ is tight.

Note that this will prove Proposition 5.1: indeed, by Proposition 4.1, $L^0(a_0 + \cdot)$ under $\mathbf{P}_n^*(\cdot \mid T(a_0) < T(0))$ converges in the sense of finite-dimensional distributions to $L^0(a_0 + \cdot)$ under $\mathcal{N}(\cdot \mid T(a_0) < T(0))$. Moreover, the jumps of $L^0(\cdot + a_0)$ are of deterministic size $1/r_n$. Since $1/r_n \to 0$, any limiting point must be continuous, see for instance [4]. Note that this reasoning could therefore be proved to show that L under $\mathbf{P}^0$ is continuous (in the space variable), a result that is difficult to prove in general (see for instance [1]).

Under the tightness assumption, the scale function w_n enjoys the following useful properties. The convexity and smoothness properties constitute one of the main reasons for making the tightness assumption.

Lemma 7.1 *For each $n \geq 1$, w_n is twice continuously differentiable and concave, its derivative w_n' is convex, $w_n(0) = 1$ and $w_n'(0) = \kappa_n$. Moreover,*

$$\sup\left\{ \frac{w_n(t)}{(1+t)^\alpha} : n \geq 1, t > 0 \right\} < +\infty$$

and finally, there exist $n_0 \geq 1$ and $t_0 > 0$ such that $w_n(t_0) \geq 2$ for all $n \geq n_0$.

Proof The smoothness of w_n follows from Theorem 3 in [9] since $f(s) = \mathbb{P}(\Lambda \geq s)$ is continuously differentiable with $|f'(0)| < +\infty$. The convexity properties follow from Theorem 2.1 in [26] since f is log-convex and $\psi_n'(0) \geq 0$. The formulas for

$w_n(0)$ and $w'_n(0)$ are well known, see for instance [25]. We now prove the two last assertions.

First of all, note that

$$\sup\left\{\frac{w_n(t)}{(1+t)^\alpha} : n \geq 1, t > 0\right\} \leq \max\left(\sup\{w_n(1) : n \geq 1\}, \sup\left\{\frac{w_n(t)}{t^{\alpha-1}} : n \geq 1, t \geq 1\right\}\right).$$

Let $\overline{\psi}$ be the Lévy exponent given by $\overline{\psi}(\lambda) = \lambda - (\alpha - 1)\mathbb{E}(1 - e^{-\lambda\Lambda})$ with corresponding scale function $\overline{w}$. Since $\kappa_n \to \alpha - 1$, $\mathbb{P}_n^0$ converges in distribution to the law of the Lévy process with Lévy exponent $\overline{\psi}$, and so it can be shown similarly as in the proof of Lemma 3.4 that $w_n(1) \to \overline{w}(1)$. The first term $\sup\{w_n(1) : n \geq 1\}$ appearing in the above maximum is therefore finite. As for the second term, since for any $n \geq 1$ we have $\kappa_n\mathbb{E}(\Lambda) = \kappa_n/(\alpha - 1) \leq 1$ by assumption, we get $\overline{\psi} \leq \psi_n$ and by monotonicity it follows that $w_n \leq \overline{w}$. Moreover, it is known that there exists a finite constant $C > 0$ such that $\overline{w}(t) \leq C/(t\overline{\psi}(1/t))$ for all $t > 0$, see Proposition III.1 or the proof of Proposition VII.10 in [2]. In particular,

$$\sup\left\{\frac{w_n(t)}{t^{\alpha-1}} : n \geq 1, t \geq 1\right\} \leq \sup\left\{\frac{\overline{w}(t)}{t^{\alpha-1}} : t \geq 1\right\} \leq C\sup\left\{\frac{t^\alpha}{\overline{\psi}(t)} : 0 < t \leq 1\right\}.$$

Since $\mathbb{P}(\Lambda \geq s) = (1 + s)^{-\alpha}$ one can check that there exists some constant $\beta > 0$ such that $\overline{\psi}(t) \sim \beta t^\alpha$ as $t \to 0$, which shows that the last upper bound is finite and proves the desired result.

To prove the last assertion of the lemma, consider n_0 large enough such that $\underline{\kappa} = \inf_{n \geq n_0}\kappa_n > 1/2$ (remember that $\kappa \to 1/(\alpha - 1) > 1$). Let $\underline{\psi}$ be the Lévy exponent given by $\underline{\psi}(\lambda) = \lambda - \underline{\kappa}\mathbb{E}(1 - e^{-\lambda\Lambda})$ and corresponding scale function $\underline{w}$. By monotonicity, we get $w_n \geq \underline{w}$ for any $n \geq n_0$, and one easily checks that $\underline{w}(\infty) = 1/(1 - \underline{\kappa}/(\alpha - 1))$. Since by choice of $\underline{\kappa}$ this last limit is strictly larger than 2, there exists $t_0 > 0$ such that $\underline{w}(t_0) \geq 2$. This proves the result. □

Since we are interested in limit theorems, we will assume in the sequel without loss of generality that there exists $t_0 > 0$ such that $w_n(t_0) \geq 2$ for all $n \geq 1$, and we henceforth fix such a t_0. We first give a short proof of Proposition 5.1 based on the two following technical results, see Theorem 13.5 in [4].

Proposition 7.2 (Case $(b - a) \vee (c - b) \leq t_0/s_n$) *For any $A > a_0$, there exist finite constants $C, \gamma \geq 0$ such that for all $n \geq 1$, $\lambda > 0$ and $a_0 \leq a < b < c \leq A$ with $(b - a) \vee (c - b) \leq t_0/s_n$,*

$$\mathbf{P}_n^*\left(\left|L^0(b) - L^0(a)\right| \wedge \left|L^0(c) - L^0(b)\right| \geq \lambda \mid T(a) < T(0)\right) \leq C\frac{(c - a)^{3/2}}{\lambda^\gamma}.$$

Proposition 7.3 (Case $b - a \geq t_0/s_n$) *For any $A > a_0$, there exist finite constants $C, \gamma \geq 0$ such that for all $n \geq 1$, $\lambda > 0$ and $a_0 \leq a < b \leq A$ with $b - a \geq t_0/s_n$,*

$$\mathbf{P}_n^*\left(\left|L^0(b) - L^0(a)\right| \geq \lambda \mid T(a) < T(0)\right) \leq C\frac{(b - a)^{3/2}}{\lambda^\gamma}.$$

Moreover, the constant γ can be taken equal to the constant γ of Proposition 7.2.

At this point, it must be said that the case $(b - a) \vee (c - b) \leq t_0/s_n$ is much harder than the case $b - a \geq t_0/s_n$. The reason is that in the former case, the bound $(c - a)^{3/2}$ cannot be achieved without taking the minimum between $|L^0(b) - L^0(a)|$ and $|L^0(c) - L^0(b)|$. Considering only one of these two terms gives a bound which can be shown to decay only linearly in $c - a$, which is not sufficient to establish tightness. This technical problem reflects that, in the well-studied context of random walks, tightness in the non-lattice case is harder than in the lattice one, where typically small oscillations, i.e., precisely when $(b - a) \vee (c - b) \leq t_0/s_n$, are significantly easier to control.

Proof of Proposition 5.1 based on Propositions 7.2 and 7.3 According to Theorem 13.5 in [4], it is enough to show that for each $A > a_0$, there exist finite constants $C, \gamma \geq 0$ and $\beta > 1$ such that for all $n \geq 1$, $\lambda > 0$ and $a_0 \leq a < b < c \leq A$,

$$\mathbf{P}_n^* \left(\left| L^0(b) - L^0(a) \right| \wedge \left| L^0(c) - L^0(b) \right| \geq \lambda \,|\, T(a_0) < T(0) \right) \leq C \frac{(c - a)^\beta}{\lambda^\gamma}. \tag{13}$$

Fix $n \geq 1$, $\lambda > 0$ and $a_0 \leq a < b < c$ and let $E = \{|L^0(b) - L^0(a)| \wedge |L^0(c) - L^0(b)| \geq \lambda\}$. Since X under $\mathbf{P}_n^*$ is spectrally positive, we have $E \subset \{L(a) > 0\}$ and so

$$\mathbf{P}_n^*(E, T(a_0) < T(0)) = \mathbf{P}_n^*(E) = \mathbf{P}_n^*(E, T(a) < T(0)).$$

Thus, Bayes formula entails

$$\mathbf{P}_n^* \left(E \,|\, T(a_0) < T(0) \right) = \frac{\mathbf{P}_n^* \left(T(a) < T(0) \right)}{\mathbf{P}_n^* \left(T(a_0) < T(0) \right)} \mathbf{P}_n^* \left(E \,|\, T(a) < T(0) \right)$$

and since $\mathbf{P}_n^* \left(T(a) < T(0) \right) \leq \mathbf{P}_n^* \left(T(a_0) < T(0) \right)$, we get

$$\mathbf{P}_n^* \left(\left| L^0(b) - L^0(a) \right| \wedge \left| L^0(c) - L^0(b) \right| \geq \lambda \,|\, T(a_0) < T(0) \right)$$
$$\leq \mathbf{P}_n^* \left(\left| L^0(b) - L^0(a) \right| \wedge \left| L^0(c) - L^0(b) \right| \geq \lambda \,|\, T(a) < T(0) \right).$$

Thus (13) follows from the previous inequality together with either Proposition 7.2 when $(b - a) \vee (c - b) \leq t_0/s_n$, or Proposition 7.3 when $b - a \geq t_0/s_n$. In the last remaining case where $c - b \geq t_0/s_n$, we derive similarly the following upper bound:

$$\mathbf{P}_n^* \left(\left| L^0(b) - L^0(a) \right| \wedge \left| L^0(c) - L^0(b) \right| \geq \lambda \,|\, T(a_0) < T(0) \right)$$
$$\leq \mathbf{P}_n^* \left(\left| L^0(c) - L^0(b) \right| \geq \lambda \,|\, T(a_0) < T(0) \right)$$
$$\leq \mathbf{P}_n^* \left(\left| L^0(c) - L^0(b) \right| \geq \lambda \,|\, T(b) < T(0) \right).$$

Proposition 7.3 then concludes the proof. □

The rest of this section is devoted to the proof of Propositions 7.2 and 7.3. Our analysis relies on an explicit expression of the law of $(L^0(b) - L^0(a), L^0(c) - L^0(b))$ under $\mathbf{P}_n^{x_0}(\cdot \mid T(a) < T(0))$. For $0 < a < b < c$ and $x_0 > 0$, we define

$$p_n^{x_0}(a) = \mathbf{P}_n^{x_0}(T(a) < T(0)), \quad p_{n,\xi}^{x_0}(a, b) = \mathbf{P}_n^{x_0}(T(b) < T(a) \mid T(a) < T(0))$$

as well as

$$p_{n,\theta}^{x_0}(a, b, c) = \mathbf{P}_n^{x_0}(T(c) < T(b) \mid T(b) < T(a) < T(0)).$$

Remember that $G_n(a)$ denotes a geometric random variable with parameter $p_n^a(a)$, and from now on we adopt the convention $\sum_1^{-1} = \sum_1^0 = 0$.

Lemma 7.4 *For any $0 < a < b < c$ and $x_0 > 0$, the random variable*

$$\left(r_n L^0(b) - r_n L^0(a), r_n L^0(c) - r_n L^0(b) \right)$$

under $\mathbf{P}_n^{x_0}(\cdot \mid T(a) < T(0))$ is equal in distribution to

$$\left(\xi_n^{x_0} + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a, \; \theta_n^{x_0} \mathbb{1}_{\{\xi_n^{x_0} \geq 0\}} + \sum_{k=1}^{N_{n,a}} \theta_{n,k}^a + \sum_{k=1}^{N_{n,b}^{x_0}} \theta_{n,k}^b \right) \tag{14}$$

where

$$N_{n,a} = \sum_{k=1}^{G_n(a)} \mathbb{1}_{\{\xi_{n,k}^a \geq 0\}} \text{ and } N_{n,b}^{x_0} = (\xi_n^{x_0})^+ + \sum_{k=1}^{G_n(a)} (\xi_{n,k}^a)^+.$$

All the random variables $\xi_n^{x_0}$, $\xi_{n,k}^a$, $\theta_n^{x_0}$, $\theta_{n,k}^a$, $\theta_{n,k}^b$ and $G_n(a)$ are independent. For any $u > 0$ and $k \geq 1$, $\theta_{n,k}^u$ is equal in distribution to ξ_n^u and $\theta_{n,k}^u$ to θ_n^u, where the laws of ξ_n^u and θ_n^u are described as follows: for any function f,

$$\mathbb{E}\left[f(\xi_n^u) \right] = (1 - p_{n,\xi}^u(a, b)) f(-1) + p_{n,\xi}^u(a, b) \mathbb{E}\left[f(G_n(b - a)) \right] \tag{15}$$

and

$$\mathbb{E}\left[f(\theta_n^u) \right] = (1 - p_{n,\theta}^u(a, b, c)) f(-1) + p_{n,\theta}^u(a, b, c) \mathbb{E}\left[f(G_n(c - b)) \right]. \tag{16}$$

Proof In the rest of the proof, we work under $\mathbf{P}_n^{x_0}(\cdot \mid T(a) < T(0))$. By definition, $r_n L^0(a)$ and $r_n L^0(b)$ is the number of visits of X to a and b in $[0, T(0)]$, respectively.

Thus, if β is the number of visits of X to b in $[0, T(a)]$ and β_k^a the number of visits of X to b between the kth and $(k+1)$st visit to a, we have

$$r_n L^0(b) - r_n L^0(a) = (\beta - 1) + \sum_{k=1}^{r_n L^0(a)-1} (\beta_k^a - 1).$$

Decomposing the path of X between successive visits to a and using the strong Markov property, one easily checks that all the random variables of the right-hand side are independent and that $r_n L^0(a)$, β_k^a and β are, respectively, equal in distribution to $1 + G_n(a)$, $\xi_n^a + 1$ and $\xi_n^{x_0} + 1$. This shows that the random variable $r_n L^0(b) - r_n L^0(a)$ is equal in distribution to $\xi_n^{x_0} + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a$.

To describe the law of $(r_n L^0(b) - r_n L^0(a), r_n L^0(c) - r_n L^0(b))$, one also needs to count the number of visits of X to c: if X visits b before a, it X may visit c before the first visit to b; it may also visit c each time it goes from a to b; finally, it may also visit c between two successive visits to b. These three different ways of visiting c are, respectively, taken into account by the terms $\theta_n^{x_0}$, $\theta_{n,k}^a$ and $\theta_{n,k}^b$.   □

The previous result readily gives the law of $(r_n L^0(b) - r_n L^0(a), r_n L^0(c) - r_n L^0(b))$ under $\mathbf{P}_n^*(\cdot \mid T(a) < T(0))$: This law can be written as

$$\left(\widetilde{\xi}_n^a + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a, \; \widetilde{\theta}_n^a \mathbb{1}_{\{\widetilde{\xi}_n^a \geq 0\}} + \sum_{k=1}^{N_{n,a}} \theta_{n,k}^a + \sum_{k=1}^{\widetilde{N}_{n,b}^a} \theta_{n,k}^b \right) \tag{17}$$

where $\widetilde{N}_{n,b}^a = (\widetilde{\xi}_n^a)^+ + \sum_{k=1}^{G_n(a)} (\xi_{n,k}^a)^+$ and the random variables $\xi_{n,k}^a$, $\theta_{n,k}^a$, $\theta_{n,k}^b$, $G_n(a)$ and $N_{n,a}$ are as described in Lemma 7.4. Moreover, these random variables are also independent from the pair $(\widetilde{\xi}_n^a, \widetilde{\theta}_n^a)$ whose distribution is given by

$$\mathbb{E}\left[f(\widetilde{\xi}_n^a, \widetilde{\theta}_n^a) \right] = \int \mathbb{E}\left[f(\xi_n^{x_0}, \theta_n^{x_0}) \right] \mathbb{P}(\chi_n^a \in dx_0) \tag{18}$$

where from now on χ_n^a denotes a random variable equal in distribution to $X(0)$ under $\mathbf{P}_n^*(\cdot \mid T(a) < T(0))$. For convenience, we will sometimes consider that χ_n^a lives on the same probability space and is independent from all the other random variables. This will for instance allow us to say that the random vector (17) conditional on $\{\chi_n^a = x_0\}$ is equal in distribution to the random vector (14).

In order to exploit (17) and prove Propositions 7.2 and 7.3, we will use a method based on controlling the moments, following similar lines as [5,6]. As Propositions 7.2 and 7.3 suggest, we need to distinguish the two cases $(b - a) \vee (c - b) \leq t_0/s_n$ and $b - a \geq t_0/s_n$ (remember that t_0 is a fixed number such that $w_n(t_0) \geq 2$ for each $n \geq 1$, see the discussion after Lemma 7.1).

In the sequel, we need to derive numerous upper bounds. The letter C then denotes constants which may change from line to line (and even within one line) but never depend on n, a, b, c, x_0 or λ. They may however depend on other variables, such as typically a_0, A or t_0.

Before starting, let us gather a few relations and properties that will be used repeatedly (and sometimes without comments) in the sequel. First, it stems from (1) that

$$\mathbf{P}_n^{x_0}(T(a) < T(0)) = \frac{W_n(a) - W_n(a - x_0)}{W_n(a)} = \frac{\int_0^{x_0} W_n'(a - u)\mathrm{d}u}{W_n(a)}, \quad 0 < x_0 \leq a.$$

Moreover, (1) $\kappa_n \leq 1$; (2) $W_n'(0) = \kappa_n s_n / r_n = \kappa_n s_n^{2-\alpha}$, $W_n \geq 0$ is increasing, $W_n' \geq 0$ is decreasing and $-W_n'' \geq 0$ is decreasing (as a consequence of Lemma 7.1 together with the identity $W_n(a) = w_n(as_n)/r_n$); (3) for every $a > 0$, the sequences $(W_n(a), n \geq 1)$ and $(W_n'(a), n \geq 1)$ are bounded away from 0 and infinity (by Lemma 3.4).

7.1 Case $(b - a) \vee (c - b) \leq t_0/s_n$

The goal of this subsection is to prove Proposition 7.2 through a series of lemmas.

Lemma 7.5 *For any $A > a_0$, there exists a finite constant C such that for any $n \geq 1$, any $a_0 \leq a \leq A$ and any $0 \leq x_0 \leq y \leq a$,*

$$j_{n,a}(x_0, y) \leq \begin{cases} Cx_0(a - y)s_n^{2-\alpha} & \text{if } x_0 \geq a/4, \\ Cx_0(a - y) & \text{if } x_0 \leq a/4 \text{ and } a/2 \leq y, \\ Cx_0 W_n'(y - x_0) & \text{if } x_0 \leq a/4 \text{ and } y \leq a/2, \end{cases} \quad (19)$$

where $j_{n,a}(x_0, y) = W_n(a - x_0)W_n(y) - W_n(y - x_0)W_n(a)$.

Proof The derivation of these three bounds is based on the following identity:

$$j_{n,a}(x_0, y) = \int_0^{a-y} W_n'(u + y)\mathrm{d}u \int_0^{x_0} W_n'(u + y - x_0)\mathrm{d}u$$

$$+ W_n(y) \int_0^{x_0} \int_0^{a-y} (-W_n'')(u + v + y - x_0)\mathrm{d}u\mathrm{d}v.$$

The term $W_n(y)$ appearing in the second term of the right-hand side is upper bounded by the finite constant $\sup_{n \geq 1} W_n(A)$ and so does not play a role as far as (19) is concerned. Assume that $x_0 \geq a/4$: then $y \geq a_0/4$ and so

$$\int_0^{a-y} W_n'(u + y)\mathrm{d}u \int_0^{x_0} W_n'(u + y - x_0)\mathrm{d}u \leq (a - y)W_n'(a_0/4)x_0 W_n'(0)$$

$$\leq C(a - y)x_0 s_n^{2-\alpha}.$$

On the other hand,

$$\int\limits_0^{x_0}\int\limits_0^{a-y}(-W_n'')(u+v+y-x_0)\mathrm{d}u\mathrm{d}v \le (a-y)\int\limits_0^{x_0}(-W_n'') \le (a-y)W_n'(0).$$

This gives the desired upper bound of the form $Cx_0 s_n^{2-\alpha}$, since $W_n'(0) \le s_n^{2-\alpha}$ and $1 \le Cx_0$ when $x_0 \ge a/4$ (writing $1 = x_0/x_0 \le 4x_0/a_0$). This proves (19) in the case $x_0 \ge a/4$. Assume now that $x_0 \le a/4 \le a/2 \le y$, so that $y \ge a_0/2$ and $y - x_0 \ge a_0/4$. Then

$$\int\limits_0^{a-y}W_n'(u+y)\mathrm{d}u\int\limits_0^{x_0}W_n'(u+y-x_0)\mathrm{d}u \le (a-y)W_n'(a_0/2)x_0 W_n'(a_0/4)$$

$$\le Cx_0(a-y)$$

and

$$\int\limits_0^{x_0}\int\limits_0^{a-y}(-W_n'')(u+v+y-x_0)\mathrm{d}u\mathrm{d}v \le \int\limits_0^{x_0}\int\limits_0^{a-y}(-W_n'')(u+v+a_0/4)\mathrm{d}u\mathrm{d}v.$$

Since $-W_n''$ is decreasing, so is the function $\varphi(u) = \int_0^{a-y}(-W_n'')(u+v+a_0/4)\mathrm{d}v$. Differentiating, this immediately shows that the function $z \mapsto z^{-1}\int_0^z \varphi$ is decreasing and since $x_0 \ge a_0$, this gives

$$\int\limits_0^{x_0}\int\limits_0^{a-y}(-W_n'')(u+v+a_0/4)\mathrm{d}u\mathrm{d}v \le x_0 a_0^{-1}\int\limits_0^{a-y}\int\limits_0^{a_0}(-W_n'')(u+v+a_0/4)\mathrm{d}u\mathrm{d}v.$$

Further, exploiting that $-W_n''$ is decreasing, we obtain

$$\int\limits_0^{x_0}\int\limits_0^{a-y}(-W_n'')(u+v+a_0/4)\mathrm{d}u\mathrm{d}v \le x_0 a_0^{-1}(a-y)\int\limits_0^{a_0}(-W_n'')(u+a_0/4)\mathrm{d}u$$

$$\le Cx_0(a-y).$$

This proves (19) in the case $x_0 \le a/4 \le a/2 \le y$. Assume now that $x_0 \le a/4$ and that $y \le a/2$: then

$$\int\limits_0^{a-y}W_n'(u+y)\mathrm{d}u\int\limits_0^{x_0}W_n'(u+y-x_0)\mathrm{d}u \le Cx_0 W_n'(y-x_0)$$

and

$$\int\limits_0^{x_0} \int\limits_0^{a-y} (-W_n'')(u+v+y-x_0)\mathrm{d}u\mathrm{d}v \le x_0 \int\limits_0^{a-y} (-W_n'')(u+y-x_0)\mathrm{d}u$$

$$\le x_0 W_n'(y-x_0).$$

This concludes the proof of (19),   □

Lemma 7.6 *For any $A > a_0$, there exists a finite constant C such that for any $n \ge 1$, any $a_0 \le a < b \le A$ and any $x_0 \le a$,*

$$1 - p_{n,\xi}^{x_0}(a,b) \le C(b-a)s_n \tag{20}$$

and for any $n \ge 1$, any $a_0 \le a < b < c \le A$ with $b - a \le t_0/s_n$ and any $x_0 \le b$,

$$1 - p_{n,\theta}^{x_0}(a,b,c) \le C(c-b)s_n. \tag{21}$$

Proof Let us first prove (20), so assume until further notice $x_0 \le a$. Let in the rest of the proof $\tau_a = \inf\{t \ge 0 : X(t) \notin (0,a)\}$: then the inclusion

$$\{T(a) < T(b), T(a) < T(0)\} \subset \{a \le X(\tau_a) < b\}$$

holds $\mathbf{P}_n^{x_0}$-almost surely and leads to

$$1 - p_{n,\xi}^{x_0}(a,b) = \frac{\mathbf{P}_n^{x_0}(T(a) < T(b), T(a) < T(0))}{\mathbf{P}_n^{x_0}(T(a) < T(0))} \le \mathbf{P}_n^{x_0}(a \le X(\tau_a) < b) \frac{W_n(a)}{\int_0^{x_0} W_n'(a-u)\mathrm{d}u}$$

$$\le Cx_0^{-1}\mathbf{P}_n^{x_0}(a \le X(\tau_a) < b).$$

Thus (20) will be proved if we can show that $\mathbf{P}_n^{x_0}(a \le X(\tau_a) < b) \le Cx_0(b-a)s_n$. Let $h_n(z) = \alpha\kappa_n s_n^{\alpha+1}(1+zs_n)^{-\alpha-1}$ be the density of the measure Π_n with respect to Lebesgue measure; note that it is decreasing and that the sequence $(h_n(z))$ is bounded for any $z > 0$. Corollary 2 in [3] gives

$$\mathbf{P}_n^{x_0}(X(\tau_a-) \in \mathrm{d}y, \Delta X(\tau_a) \in \mathrm{d}z) = u_{n,a}(x_0,y)h_n(z)\mathrm{d}z\mathrm{d}y, \quad y \le a \le y + z, \tag{22}$$

where $\Delta X(t) = X(t) - X(t-)$ for any $t \ge 0$ and

$$u_{n,a}(x_0,y) = \frac{W_n(a-x_0)W_n(y)}{W_n(a)} - W_n(y-x_0)\mathbb{1}_{\{y \ge x_0\}},$$

so it follows that

$$\mathbf{P}_n^{x_0}\left(a \leq X(\tau_a) < b\right) = \int \mathbb{1}_{\{y<a,a-y\leq z<b-y\}} u_{n,a}(x_0, y) h_n(z) dz dy$$

$$\leq (b-a) \int_0^a u_{n,a}(x_0, y) h_n(a - y) dy.$$

Hence (20) will be proved if we can show that $\int_0^a u_{n,a}(x_0, y) h_n(a-y) dy \leq Cx_0 s_n$. We have by definition of $u_{n,a}$

$$\int_0^a u_{n,a}(x_0, y) h_n(a - y) dy = \frac{W_n(a - x_0)}{W_n(a)} \int_0^{x_0} W_n(y) h_n(a - y) dy$$

$$+ \int_{x_0}^a \left(\frac{W_n(a - x_0) W_n(y)}{W_n(a)} - W_n(y - x_0)\right) h_n(a-y) dy$$

which readily implies

$$\int_0^a u_{n,a}(x_0, y) h_n(a - y) dy \leq C W_n(a - x_0) \int_0^{x_0} h_n(a - y) dy$$

$$+ C \int_{x_0}^a j_{n,a}(x_0, y) h_n(a - y) dy \tag{23}$$

with $j_{n,a}(x_0, y) = W_n(a - x_0) W_n(y) - W_n(y - x_0) W_n(a)$ as in Lemma 7.5. We will show that each term of the above right-hand side is upper bounded by a term of the form $Cx_0 s_n$. Let us focus on the first term, so we want to show that

$$W_n(a - x_0) \int_0^{x_0} h_n(a - y) dy \leq Cx_0 s_n.$$

Assume first that $x_0 \leq a/2$, then $a - y \geq a - x_0 \geq a/2 \geq a_0/2$ for any $y \leq x_0$ which gives

$$W_n(a - x_0) \int_0^{x_0} h_n(a - y) dy \leq W_n(A) x_0 h_n(a_0/2) \leq Cx_0 \leq Cx_0 s_n.$$

Assume now $x_0 \geq a/2$: Since h_n is the density of Π_n, we have

$$W_n(a - x_0) \int_0^{x_0} h_n(a - y)\mathrm{d}y \leq W_n(a - x_0)\Pi_n((a - x_0, \infty)) = \kappa_n s_n \frac{w_n((a - x_0)s_n)}{(1 + (a - x_0)s_n)^\alpha}.$$

Since $1 \leq Cx_0$ (because $x_0 \geq a_0/2$), the desired upper bound of the form $Cx_0 s_n$ follows from Lemma 7.1. We now control the second term of the right-hand side in (23), i.e., we have to show that

$$\int_{x_0}^a j_{n,a}(x_0, y)h_n(a - y)\mathrm{d}y \leq Cx_0 s_n.$$

In the case $x_0 \geq a/4$, the first bound in (19) gives

$$\int_{x_0}^a j_{n,a}(x_0, y)h_n(a - y)\mathrm{d}y \leq Cx_0 s_n^{2-\alpha} \int_{x_0}^a \frac{(a - y)s_n^{\alpha+1}}{(1 + (a - y)s_n)^{\alpha+1}}\mathrm{d}y$$

$$= Cx_0 s_n \int_0^{(a-x_0)s_n} \frac{y}{(1 + y)^{\alpha+1}}\mathrm{d}y \leq Cx_0 s_n.$$

Assume from now on that $x_0 \leq a/4$ and decompose the interval $[x_0, a]$ into the union $[x_0, a/2] \cup [a/2, a]$. For $[a/2, a]$, (19) gives

$$\int_{a/2}^a j_{n,a}(x_0, y)h_n(a - y)\mathrm{d}y \leq Cx_0 \int_{a/2}^a \frac{(a - y)s_n^{\alpha+1}}{(1 + (a - y)s_n)^{\alpha+1}}\mathrm{d}y \leq Cx_0 s_n^{\alpha-1} \leq Cx_0 s_n$$

since $\alpha - 1 < 1$. For $[x_0, a/2]$, (19) gives, using $a - y \geq a_0/2$ when $y \leq a/2$,

$$\int_{x_0}^{a/2} j_{n,a}(x_0, y)h_n(a - y)\mathrm{d}y \leq Cx_0 h_n(a_0/2) \int_{x_0}^a W_n'(y - x_0)\mathrm{d}y \leq Cx_0 \leq Cx_0 s_n.$$

This finally concludes the proof of (20), which we use to derive (21). Assume from now on that $x_0 \leq b$, we have by definition

$$\begin{aligned}
1 - p_{n,\theta}^{x_0}(a, b, c) &= \frac{\mathbf{P}_n^{x_0}(T(b) < T(c), T(b) < T(a) < T(0))}{\mathbf{P}_n^{x_0}(T(b) < T(a) < T(0))} \\
&\leq \frac{\mathbf{P}_n^{x_0}(T(b) < T(c), T(b) < T(0))}{\mathbf{P}_n^{x_0}(T(b) < T(a) < T(0))} \\
&= \frac{\mathbf{P}_n^{x_0}(T(b) < T(0))}{\mathbf{P}_n^{x_0}(T(b) < T(a) < T(0))}\left(1 - p_{n,\xi}^{x_0}(b, c)\right).
\end{aligned}$$

Since $\mathbf{P}_n^z(T(b) < T(0)) = \int_0^z \varphi$ where $\varphi(u) = W_n'(b-u)/W_n(b)$ is increasing, it is readily shown by differentiating that $z \in [0, b] \mapsto z^{-1}\mathbf{P}_n^z(T(b) < T(0))$ is also increasing. Thus, for $x_0 \le b$, we obtain

$$\mathbf{P}_n^{x_0}(T(b) < T(0)) \le x_0 b^{-1}\mathbf{P}_n^b(T(b) < T(0)) \le Cx_0.$$

In combination with (20) and the fact that $\mathbf{P}_n^{x_0}(X(\tau_a) \ge b) \le \mathbf{P}_n^{x_0}(T(b) < T(a) < T(0))$, this entails

$$1 - p_{n,\theta}^{x_0}(a, b, c) \le C(c-b)s_n \frac{x_0}{\mathbf{P}_n^{x_0}(X(\tau_a) \ge b)}.$$

Hence (21) will be proved if we show that $x_0 \le C\mathbf{P}_n^{x_0}(X(\tau_a) \ge b)$. If follows from (22) that

$$\mathbf{P}_n^{x_0}(X(\tau_a) \ge b) = \int_0^a u_{n,a}(x_0, y)\kappa_n s_n^\alpha \mathbb{P}(\Lambda \ge (b-y)s_n)\mathrm{d}y$$

and because $\mathbb{P}(\Lambda \ge u) = (1+s)^{-\alpha}$, it can be checked that $\mathbb{P}(\Lambda \ge u+v) \ge \mathbb{P}(\Lambda \ge u)\mathbb{P}(\Lambda \ge v)$ for any $u, v > 0$, so that

$$\mathbf{P}_n^{x_0}(X(\tau_a) \ge b) \ge \mathbb{P}(\Lambda \ge (b-a)s_n) \int_0^a u_{n,a}(x_0, y)\kappa_n s_n^\alpha \mathbb{P}(\Lambda \ge (a-y)s_n)\mathrm{d}y.$$

In view of (22), this last integral is equal to $\mathbf{P}_n^{x_0}(X(\tau_a) \ge a) = \mathbf{P}_n^{x_0}(T(a) < T(0))$ so finally, using $(b-a)s_n \le t_0$ we get

$$\frac{x_0}{\mathbf{P}_n^{x_0}(X(\tau_a) \ge b)} \le \frac{x_0 W_n(a)}{\mathbb{P}(\Lambda \ge t_0)(W_n(a) - W_n(a - x_0))} \le C$$

which proves (21). $\qquad\qquad\square$

Lemma 7.7 *There exists a finite constant C such that for all $a_0 \le a < b$ and all $n \ge 1$,*

$$\left|\mathbb{E}\left(\xi_n^a\right)\right| \le C(b-a)s_n/(r_n W_n(a)).$$

Proof Starting from a, X (under $\mathbf{P}_n^a$) makes $1 + G_n(a)$ visits to a. Decomposing the path $(X(t), 0 \le t \le T(0))$ between successive visits to a, one gets

$$\mathbf{P}_n^a(T(b) > T(0)) = \mathbb{E}\left[\left\{\mathbf{P}_n^a(T(a) < T(b) \mid T(a) < T(0))\right\}^{G_n(a)}\right]$$

$$= \mathbb{E}\left[(1 - p_{n,\xi}^a(a, b))^{G_n(a)}\right].$$

By definition, the left-hand side is equal to $1 - p_n^a(b)$, so integrating on $G_n(a)$ gives

$$\mathbb{E}\left[(1 - p_{n,\xi}^a(a,b))^{G_n(a)}\right] = \frac{1 - p_n^a(a)}{1 - (1 - p_{n,\xi}^a(a,b))p_n^a(a)} = 1 - p_n^a(b)$$

which gives

$$1 - p_{n,\xi}^a(a,b) = \frac{W_n(b-a)W_n(a) - W_n(b)W_n(0)}{W_n(b-a)(W_n(a) - W_n(0))}. \tag{24}$$

Let $p_n = p_n^{b-a}(b-a)$ and $p_{n,\xi} = p_{n,\xi}^a(a,b)$: We have

$$\mathbb{E}\left(\xi_n^a\right) = p_{n,\xi} - 1 + p_{n,\xi}\mathbb{E}(G_n(b-a)) = p_{n,\xi} - 1 + p_{n,\xi}\frac{p_n}{1-p_n} = \frac{p_{n,\xi}}{1-p_n} - 1.$$

Plugging in (1) and (24) gives after some computation

$$\mathbb{E}\left(\xi_n^a\right) = \frac{W_n(b) - W_n(a) - (W_n(b-a) - W_n(0))}{W_n(a) - W_n(0)} = \frac{\int_0^a \int_0^{b-a} W_n''(u+v)\mathrm{d}u\mathrm{d}v}{\int_0^a W_n'}.$$

Since $W_n' \geq 0$ and $W_n'' \leq 0$, this gives

$$\left|\mathbb{E}\left(\xi_n^a\right)\right| = \frac{\int_0^a \int_0^{b-a} (-W_n'')(u+v)\mathrm{d}u\mathrm{d}v}{\int_0^a W_n'} \tag{25}$$

and since W_n' is convex, we get

$$\left|\mathbb{E}\left(\xi_n^a\right)\right| \leq (b-a)\frac{\int_0^a (-W_n'')}{\int_0^a W_n'} = \frac{(b-a)W_n'(0)}{W_n(a)}\frac{W_n(a)(W_n'(0) - W_n'(a))}{W_n'(0)(W_n(a) - W_n(0))}.$$

Since $W_n'(0) \leq s_n/r_n$ and

$$\frac{W_n(a)(W_n'(0) - W_n'(a))}{W_n'(0)(W_n(a) - W_n(0))} \leq \frac{W_n(a_0)}{W_n(a_0) - W_n(0)} \leq C,$$

the proof is complete.   □

To control the higher moments of ξ_n^a and also the moments of the θ's, we introduce the following constants:

$$C_i = \sup\left\{\frac{\mathbb{E}\left((G_n(\delta))^i\right)}{\delta s_n} : n \geq 1, 0 \leq \delta \leq t_0/s_n\right\}, \quad i \geq 1. \tag{26}$$

Lemma 7.8 *For any integer $i \geq 1$, the constant C_i is finite.*

Proof Using the concavity of w_n, one gets $w_n(\delta s_n) \leq w_n(0) + \delta s_n w'_n(0) \leq 1 + \delta s_n$ since $w_n(0) = 1$ and $w'_n(0) = \kappa_n \leq 1$ by Lemma 7.1. Hence

$$p_n^\delta(\delta) = \frac{w_n(\delta s_n) - w_n(0)}{w_n(\delta s_n)} \leq \frac{\delta s_n}{1 + \delta s_n} \leq \min\left(\delta s_n, \frac{t_0}{1 + t_0}\right)$$

where the last inequality holds for $\delta s_n \leq t_0$. In particular, for any $i \geq 1$, we have

$$\frac{\mathbb{E}\left((G_n(\delta))^i\right)}{\delta s_n} = \frac{\mathbb{E}\left((G_n(\delta))^i\right)}{p_n^\delta(\delta)} \frac{p_n^\delta(\delta)}{\delta s_n} \leq \sup_{0 \leq p \leq \varepsilon} \left(p^{-1}\mathbb{E}\left(G_p^i\right)\right)$$

where $\varepsilon = t_0/(1 + t_0) < 1$ and G_p is a geometric random variable with parameter p. It is well known that

$$\mathbb{E}(G_p^i) = \frac{p P_{i-1}(p)}{(1 - p)^i}$$

where P_i is the polynomial $P_i(p) = \sum_{k=0}^i T_{k,i} p^i$ with $T_{k,i} \geq 1$ the Eulerian numbers. Since P_i satisfies $P_i(0) = 1$, one easily sees that for any $\varepsilon < 1$,

$$\sup_{0 \leq p \leq \varepsilon} \left(p^{-1}\mathbb{E}\left(G_p^i\right)\right) < +\infty$$

which achieves the proof. $\square$

Lemma 7.9 *For any $A > a_0$ and $i \geq 1$, there exists a finite constant C such that for all $n \geq 1$ and all $a_0 \leq a < b < c \leq A$ with $(b - a) \vee (c - b) \leq t_0/s_n$*

$$\max\left(\mathbb{E}\left(|\xi_n^a|^i\right), \mathbb{E}\left(|\theta_n^a|^i\right), \mathbb{E}\left(|\theta_n^b|^i\right)\right) \leq C(c - a)s_n. \tag{27}$$

Proof The results for ξ_n^a and θ_n^a are direct consequences of (20) and (21) and the finiteness of C_i: Indeed, using these two results, we have for instance for ξ_n^a

$$\mathbb{E}\left(|\xi_n^a|^i\right) = 1 - p_{n,\xi}^a(a, b) + p_{n,\xi}^a(a, b)\mathbb{E}\left((G_n(b-a))^i\right) \leq C(b - a)s_n + C_i(b - a)s_n$$

and similarly for θ_n^a. The result for θ_n^b is also straightforward because

$$\begin{aligned}
p_{n,\theta}^b(a, b, c) &= \mathbf{P}_n^b\left(T(c) < T(b) \mid T(b) < T(a) < T(0)\right) \\
&= \mathbf{P}_n^b\left(T(c) < T(b) \mid T(b) < T(a)\right) \\
&= p_{n,\xi}^{b-a}(c - a, b - a).
\end{aligned}$$

and so the result follows similarly as for ξ_n^a. $\square$

Recall the random variables $\widetilde{\xi}_n^a$ and $\widetilde{\theta}_n^a$ defined in (18).

Lemma 7.10 *For any $A > a_0$, there exists a finite constant C such that for all $n \geq 1$ and all $a_0 \leq a < b < c \leq A$ with $(b - a) \vee (c - b) \leq t_0/s_n$,*

$$\mathbb{E}\left(|\widetilde{\xi}_n^a|^i\right) \leq C(b - a)s_n \text{ and } \mathbb{E}\left(|\widetilde{\theta}_n^a|^i\right) \leq C(c - a)s_n. \tag{28}$$

Proof Combining the two definitions (15) and (18), we obtain

$$\mathbb{E}\left(|\widetilde{\xi}_n^a|^i\right) = \int_0^\infty \left(1 - p_{n,\xi}^{x_0}(a, b) + p_{n,\xi}^{x_0}(a, b)\mathbb{E}\left((G_n(b - a))^i\right)\right)\mathbb{P}(\chi_n^a \in \mathrm{d}x_0)$$

$$\leq C(b - a)s_n + \mathbb{P}(a \leq \chi_n^a \leq b) + C_i(b - a)s_n$$

using (20) to obtain the inequality. We obtain similarly for $\widetilde{\theta}_n^a$, using (21) instead of (20),

$$\mathbb{E}\left(|\widetilde{\theta}_n^a|^i\right) \leq C(c - b)s_n + \mathbb{P}(b \leq \chi_n^a \leq c) + C_i(c - b)s_n.$$

Thus, the result will be proved if we can show that $\mathbb{P}(a \leq \chi_n^a \leq c) \leq C(c - a)s_n$; remember that χ_n^a is by definition equal in distribution to $X(0)$ under $\mathbf{P}_n^*(\cdot \mid T(a) < T(0))$. Because X is spectrally positive and $a \leq A$, it holds that

$$\mathbf{P}_n^*(a < X(0) \leq c \mid T(a) < T(0)) = \frac{\mathbf{P}_n^*(a < X(0) \leq c)}{\mathbf{P}_n^*(T(a) < T(0))} \leq \frac{\mathbf{P}_n^*(a < X(0) \leq c)}{\mathbf{P}_n^*(T(A) < T(0))}.$$

Since $\mathbf{P}_n^*(T(A) < T(0)) = \mathbf{P}_n^0(T(A) < T(0))$ by Lemma 2.1, Lemma 3.6 implies that

$$\inf_{n \geq 1}\left(r_n\mathbf{P}_n^*(T(A) < T(0))\right) > 0$$

which leads to $\mathbf{P}_n^*(a < X(0) \leq c \mid T(a) < T(0)) \leq Cr_n\mathbf{P}_n^*(a < X(0) \leq c)$. We have by definition

$$r_n\mathbf{P}_n^*(a < X(0) \leq c) = (\alpha - 1)r_n\int_{as_n}^{cs_n}\frac{\mathrm{d}u}{(1 + u)^\alpha} \leq C(c - a) \leq C(c - a)s_n$$

which achieves the proof. ∎

To control the sum of i.i.d. random variables, we will repeatedly use the following simple combinatorial lemma. In the sequel for $I \in \mathbb{N}$ and $\beta \in \mathbb{N}^I$ note $|\beta| = \sum \beta_i$ and $\|\beta\| = \sum i\beta_i$.

Lemma 7.11 *Let (Y_k) be i.i.d. random variables with common distribution Y. Then for any even integer $I \geq 0$ and any $K \geq 0$,*

$$\mathbb{E}\left[\left(\sum_{k=1}^{K} Y_i\right)^I\right] \leq I^I \sum_{\beta \in \mathbb{N}^I : \|\beta\| = I} K^{|\beta|} \prod_{i=1}^{I} \left|\mathbb{E}\left(Y^i\right)\right|^{\beta_i}.$$

Proof We have $\mathbb{E}\left[(Y_1 + \cdots + Y_K)^I\right] = \sum_{1 \leq k_1, \ldots, k_I \leq K} \mathbb{E}(Y_{k_1} \ldots Y_{k_I})$. Since the (Y_k)'s are i.i.d., we have $\mathbb{E}\left(Y_{k_1} \ldots Y_{k_I}\right) = m_1^{\beta_1} \ldots m_I^{\beta_I}$ with $m_i = \mathbb{E}(Y^i)$ and β_i the number of i-tuples of k, i.e., β_1 is the number of singletons, β_2 the number of pairs, etc … Since I is even, this leads to

$$\mathbb{E}\left[(Y_1 + \cdots + Y_K)^I\right] \leq \sum_{0 \leq \beta_1, \ldots, \beta_I \leq K : \|\beta\| = I} A_{I,K}(\beta) |m_1|^{\beta_1} \ldots |m_I|^{\beta_I}$$

with $A_{I,K}(\beta)$ the number of I-tuples $k \in \{1, \ldots, K\}^I$ with exactly β_i i-tuples for each $i = 1, \ldots, I$. There are $K(K-1) \ldots (K - (|\beta| - 1))$ different ways of choosing the $|\beta|$ different values taken by k, thus $A_{I,K}(\beta) = K(K-1) \ldots (K - (|\beta| - 1)) \times B(I, |\beta|)$ with $B(i, a)$ the number of ways of assigning i objects into a different boxes in such a way that no box is empty, so that $A_{I,K}(\beta) \leq K^{|\beta|} I^{|\beta|} \leq K^{|\beta|} I^I$ since $|\beta| \leq \|\beta\| = I$. $\qquad\square$

In the sequel, we will use the inequality

$$\mathbb{E}\left((G_n(a))^i\right) \leq i!(r_n W_n(a))^i, \quad i \geq 1, a > 0,$$

which comes from the fact that $G_n(a)$ is stochastically dominated by an exponential random variable with parameter $1 - p_n^a(a) = 1/(r_n W_n(a))$. We now use the previous bounds on the moments to control the probability

$$\mathbb{P}\left(\left|\widetilde{\xi}_n^a + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a\right| \geq \lambda r_n\right).$$

We will see that it achieves a linear bound (in $b - a$) which justifies the need of the min later on.

Lemma 7.12 *For any $A > a_0$ and any even integer $I \geq 2$, there exists a finite constant C such that for all $n \geq 1$, all $\lambda > 0$ and all $a_0 \leq a < b \leq A$ with $b - a \leq t_0/s_n$,*

$$\mathbb{P}\left(\left|Y + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a\right| \geq \lambda r_n\right) \leq C \frac{b - a}{\lambda^I} s_n r_n^{-I/2}$$

where Y is any random variable equal in distribution either to $\widetilde{\xi}_n^a$ or to $G_n(b - a)$.

Proof Using first the triangular inequality and then Markov inequality gives

$$\mathbb{P}\left(\left| Y + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a \right| \ge \lambda r_n \right) \le (2/\lambda r_n)^I \left(\mathbb{E}\left(Y^I\right) + \mathbb{E}\left[\left(\sum_{k=1}^{G_n(a)} \xi_{n,k}^a \right)^I \right] \right).$$

Using the independence between $G_n(a)$ and $(\xi_{n,k}^a, k \ge 1)$ together with Lemma 7.11 gives

$$\mathbb{E}\left[\left(\sum_{k=1}^{G_n(a)} \xi_{n,k}^a \right)^I \right] \le C \sum_{\beta \in \mathbb{N}^I : \|\beta\| = I} \mathbb{E}\left((G_n(a))^{|\beta|} \right) \prod_{i=1}^{I} \left| \mathbb{E}\left((\xi_n^a)^i \right) \right|^{\beta_i}$$

$$\le C \sum_{\beta \in \mathbb{N}^I : \|\beta\| = I} (r_n W_n(a))^{|\beta|} \prod_{i=1}^{I} \left| \mathbb{E}\left((\xi_n^a)^i \right) \right|^{\beta_i}.$$

Lemma 7.7 gives the bound

$$\prod_{i=1}^{I} \left| \mathbb{E}\left((\xi_n^a)^i \right) \right|^{\beta_i} \le C \left((b-a)s_n / (r_n W_n(a)) \right)^{\beta_1} \prod_{i=2}^{I} ((b-a)s_n)^{\beta_i}$$

$$= C((b-a)s_n)^{|\beta|} (r_n W_n(a))^{-\beta_1} \le C(b-a)s_n (r_n W_n(a))^{-\beta_1}$$

where $((b-a)s_n)^{|\beta|} \le C(b-a)s_n$ follows from the fact that $(b-a)s_n \le t_0$ while $|\beta| \ge 1$. Using (28) for the case $Y = \widetilde{\xi}_n^a$ and the finiteness of C_I for the case $Y = G_n(b-a)$, one can write $\mathbb{E}(Y^I) \le C(b-a)s_n$, which gives

$$\mathbb{P}\left(\left| Y + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a \right| \ge \lambda r_n \right) \le C(\lambda r_n)^{-I} (b-a)s_n \left(1 + \sum_{\beta \in \mathbb{N}^I : \|\beta\| = I} (r_n W_n(a))^{|\beta| - \beta_1} \right).$$

But

$$|\beta| - \beta_1 = \sum_{i=2}^{I} \beta_i \le \sum_{i=2}^{I} (i/2)\beta_i = \frac{1}{2} \left(\|\beta\| - \beta_1 \right) \le I/2 \qquad (29)$$

and $r_n W_n(a) \ge 1$ so $(r_n W_n(a))^{|\beta| - \beta_1} \le (r_n W_n(a))^{I/2}$, which proves the result. □

Lemma 7.13 *For any $i \ge 1$ and $A > 0$, it holds that*

$$\sup \left\{ r_n^{-i} \mathbb{E}\left((N_{n,a})^i \right) : n \ge 1, 0 < a < b \le A \right\} < +\infty$$

and

$$\sup \left\{ r_n^{-i} \mathbb{E}\left((\widetilde{N}_{n,b}^a)^i \right) : n \ge 1, 0 < a < b < c \le A, b - a \le t_0 / s_n \right\} < +\infty.$$

Proof The result on $N_{n,a}$ comes from the following inequality $\mathbb{E}((N_{n,a})^i) \leq \mathbb{E}((G_n(a))^i)$. For $\widetilde{N}^a_{n,b}$, we use the fact that $|\widetilde{\xi}^a_n|$ is stochastically dominated by $1 + G_n(b-a)$ (since for any $x_0 > 0$, $|\xi^{x_0}_n|$ is), thus

$$\mathbb{E}\left((\widetilde{N}^a_{n,b})^i\right) \leq \mathbb{E}\left(\left(\sum_{k=1}^{G_n(a)+1} (1 + G_{n,k}(b-a))\right)^i\right)$$

with $(G_{n,k}(b-a), k \geq 1)$ i.i.d. with common distribution $G_n(b-a)$, independent of $G_n(a)$. Thus, Lemma 7.11 gives

$$\mathbb{E}\left((\widetilde{N}^a_{n,b})^i\right) \leq C \sum_{\beta \in \mathbb{N}^i : \|\beta\| = i} \mathbb{E}\left((1 + G_n(a))^{|\beta|}\right) \prod_{k=1}^{i} \left[\mathbb{E}\left((1 + G_n(b-a))^k\right)\right]^{\beta_k}.$$

Since $G_n(a)$ is stochastically dominated by an exponential random variable with parameter $1 - p^a_n(a) = 1/(r_n W_n(a))$ and $G_n(b-a)$ is integer valued, so that $(1 + G_n(b-a))^k \leq (1 + G_n(b-a))^i$ for any $1 \leq k \leq i$, we get, using that $|\beta| \leq i$ and that all quantities are greater than 1,

$$\mathbb{E}\left((\widetilde{N}^a_{n,b})^i\right) \leq C\mathbb{E}\left((1 + Er_n W_n(a))^i\right) \left[\mathbb{E}\left((1 + G_n(b-a))^i\right)\right]^i$$

where E is a mean-1 exponential random variable. Using that for each $1 \leq k \leq i$

$$\mathbb{E}\left((G_n(b-a))^k\right) \leq \mathbb{E}\left((G_n(b-a))^i\right) \leq C_i(b-a)s_n \leq C_i t_0,$$

one gets

$$\sup\left\{\left[\mathbb{E}\left((1 + G_n(b-a))^i\right)\right]^i : n \geq 1, b-a \leq t_0/s_n\right\} < +\infty.$$

Together with the inequality

$$\mathbb{E}\left((1 + Er_n W_n(a))^i\right) \leq \mathbb{E}\left((1 + Er_n W_n(A))^i\right) \leq Cr^i_n$$

this concludes the proof. $\square$

We can now prove Proposition 7.2. Remember that we must find constants C and $\gamma > 0$ such that

$$\mathbf{P}^*_n\left(\left|L^0(c) - L^0(b)\right| \wedge \left|L^0(b) - L^0(a)\right| \geq \lambda \mid T(a) < T(0)\right) \leq C\frac{(c-a)^{3/2}}{\lambda^\gamma}$$

uniformly in $n \geq 1$, $\lambda > 0$ and $a_0 \leq a < b < c \leq A$ with $(b-a) \vee (c-b) \leq t_0/s_n$.

Proof of Proposition 7.2 Fix four even integers I_1, I_2, I_3, I_4. By (17),

$$
\mathbf{P}_n^* \left(\left| L^0(c) - L^0(b) \right| \wedge \left| L^0(b) - L^0(a) \right| \geq \lambda \mid T(a) < T(0) \right)
$$

$$
= \mathbb{P} \left(\left| \widetilde{\xi}_n^a + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a \right| \wedge \left| \widetilde{\theta}_n^a \mathbb{1}_{\{\widetilde{\xi}_n^a \geq 0\}} + \sum_{k=1}^{N_{n,a}} \theta_{n,k}^a + \sum_{k=1}^{\widetilde{N}_{n,b}^a} \theta_{n,k}^b \right| \geq \lambda_n \right)
$$

with $\lambda_n = \lambda r_n$. Let $\mathscr{F}$ be the σ-algebra generated by χ_n^a, $G_n(a)$, $\widetilde{\xi}_n^a$ and the $(\xi_{n,k}^a, k \geq 1)$. Then, the above probability is equal to

$$
\mathbb{E} \left\{ \pi \; ; \; \left| \widetilde{\xi}_n^a + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a \right| \geq \lambda_n \right\}
$$

with π the random variable

$$
\pi = \mathbb{P} \left(\left| \widetilde{\theta}_n^a \mathbb{1}_{\{\widetilde{\xi}_n^a \geq 0\}} + \sum_{k=1}^{N_{n,a}} \theta_{n,k}^a + \sum_{k=1}^{\widetilde{N}_{n,b}^a} \theta_{n,k}^b \right| \geq \lambda_n \;\Big|\; \mathscr{F} \right) \leq \widetilde{\pi} + \pi_a + \pi_b
$$

where

$$
\widetilde{\pi} = \mathbb{P} \left(\left| \widetilde{\theta}_n^a \mathbb{1}_{\{\widetilde{\xi}_n^a \geq 0\}} \right| \geq \lambda_n/3 \;\Big|\; \mathscr{F} \right) = \mathbb{1}_{\{\widetilde{\xi}_n^a \geq 0\}} \mathbb{P} \left(\left| \widetilde{\theta}_n^a \right| \geq \lambda_n/3 \;\Big|\; \chi_n^a \right),
$$

$$
\pi_a = \mathbb{P} \left(\left| \sum_{k=1}^{N_{n,a}} \theta_{n,k}^a \right| \geq \lambda_n/3 \;\Big|\; N_{n,a} \right) \text{ and } \pi_b = \mathbb{P} \left(\left| \sum_{k=1}^{\widetilde{N}_{n,b}^a} \theta_{n,k}^b \right| \geq \lambda_n/3 \;\Big|\; \widetilde{N}_{n,b}^a \right).
$$

The two terms π_a and π_b can be dealt with very similarly. Fix $u = a$ or b, and denote by N_u the random variable $N_{n,a}$ if $u = a$ or $\widetilde{N}_{n,b}^a$ if $u = b$. With this notation, $(\theta_{n,k}^u, k \geq 1)$ are i.i.d. and independent from N_u, so that Markov inequality and Lemma 7.11 give

$$
\pi_u \leq (3I_1/\lambda_n)^{I_1} \sum_{\beta \in \mathbb{N}^{I_1} : \|\beta\| = I_1} N_u^{|\beta|} \prod_{i=1}^{I_1} \left| \mathbb{E} \left(\left(\theta_n^u \right)^i \right) \right|^{\beta_i}.
$$

By (27),

$$
\prod_{i=1}^{I_1} \left| \mathbb{E} \left(\left(\theta_n^u \right)^i \right) \right|^{\beta_i} \leq C((c-a)s_n)^{|\beta|} \leq C(c-a)s_n
$$

since $1 \leq |\beta| \leq I_1$ and $(c - a)s_n \leq t_0$. Since N_u is integer valued, it holds that $N_u^{|\beta|} \leq N_u^{I_1}$ and finally this gives

$$\pi_u \leq C\lambda^{-I_1}(N_u/r_n)^{I_1}(c - a)s_n.$$

Applying Cauchy–Schwarz inequality yields

$$\mathbb{E}\left\{\pi_u \; ; \; \left|\widetilde{\xi}_n^a + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a\right| \geq \lambda_n\right\} \leq C\lambda^{-I_1}(c - a)s_n\sqrt{\mathbb{E}\left((N_u/r_n)^{2I_1}\right)\mathbb{P}\left(\left|\widetilde{\xi}_n^a + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a\right| \geq \lambda_n\right)}$$

and finally, Lemma 7.12 with $Y = \widetilde{\xi}_n^a$ gives, together with Lemma 7.13,

$$\mathbb{E}\left\{\pi_u \; ; \; \left|\widetilde{\xi}_n^a + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a\right| \geq \lambda_n\right\} \leq C\frac{(c - a)^{3/2}}{\lambda^{I_1+I_2/2}}s_n^{3/2}r_n^{-I_2/4}.$$

It remains to control the term $\widetilde{\pi}$: in $\{\widetilde{\xi}_n^a \geq 0\}$, $\widetilde{\xi}_n^a$ is equal in distribution to $G_n(b - a)$ and is independent of everything else; thus, we have

$$\mathbb{E}\left\{\widetilde{\pi} \; ; \; \left|\widetilde{\xi}_n^a + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a\right| \geq \lambda_n\right\} = \mathbb{E}\left\{\mathbb{P}\left(|\widetilde{\theta}_n^a| \geq \lambda_n/3 \mid \chi_n^a\right) \; ; \; \left|\widetilde{\xi}_n^a + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a\right| \geq \lambda_n, \widetilde{\xi}_n^a \geq 0\right\}$$

$$\leq C\lambda_n^{-I_3}\mathbb{E}\left\{\mathbb{E}\left(|\widetilde{\theta}_n^a|^{I_3} \mid \chi_n^a\right) \; ; \; \left|G_n(b - a) + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a\right| \geq \lambda_n\right\}.$$

Since $\mathbb{E}(|\widetilde{\theta}_n^a|^{I_3} \mid \chi_n^a)$ is independent of $G_n(b - a) + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a$, we get

$$\mathbb{E}\left\{\widetilde{\pi} \; ; \; \left|\widetilde{\xi}_n^a + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a\right| \geq \lambda_n\right\} \leq C\lambda_n^{-I_3}\mathbb{E}\left(|\widetilde{\theta}_n^a|^{I_3}\right)\mathbb{P}\left(\left|G_n(b - a) + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a\right| \geq \lambda_n\right)$$

$$\leq C\lambda_n^{-I_3-I_4}(c - a)^2s_n^2r_n^{-I_4/2}$$

where the second inequality follows using (28) and Lemma 7.12 with $Y = G_n(b - a)$. Since $(c - a)s_n \leq t_0$, we have $((c - a)s_n)^2 \leq C((c - a)s_n)^{3/2}$ and finally, gathering the previous inequalities, one sees that we have derived the bound

$$\mathbf{P}_n^*\left(\left|L^0(c) - L^0(b)\right| \wedge \left|L^0(b) - L^0(a)\right| \geq \lambda \mid T(a) < T(0)\right)$$
$$\leq C(c - a)^{3/2}s_n^{3/2}\left(\lambda^{-I_1-I_2/2}r_n^{-I_2/4} + \lambda^{-I_3-I_4}r_n^{-I_4/2}\right).$$

Now choose I_2 and I_4 large enough such that both sequences $(s_n^{3/2}r_n^{-I_2/4})$ and $(s_n^{3/2}r_n^{-I_4/2})$ are bounded: This is possible since for any $\beta \in \mathbb{R}$, $s_nr_n^{-\beta} = s_n^{1-\beta(\alpha-1)}$. Moreover, choose I_2 not only even but a multiple of 4. Then, once I_2 and I_4 are fixed, choosing I_1 and I_3 in such a way that $I_1 + I_2/2 = I_3 + I_4$ concludes the proof. $\qquad\square$

7.2 Case $b - a \geq t_0/s_n$

We now consider the simpler case $b - a \geq t_0/s_n$ and prove Proposition 7.3.

Lemma 7.14 *For any $i \geq 1$, there exists a finite constant C such that for all $n \geq 1$ and all $0 < a < b$ such that $b - a \geq t_0/s_n$,*

$$\mathbb{E}\left(|\widetilde{\xi}_n^a|^i\right) \leq C(r_n W_n(b-a))^i.$$

Proof In view of (18), it is enough to show that $\mathbb{E}\left(|\xi_n^{x_0}|^i\right) \leq C(r_n W_n(b-a))^i$ for every $x_0 > 0$. Since $b - a \geq t_0/s_n$, exploiting the monotonicity of w_n gives

$$p_n^{b-a}(b-a) = 1 - \frac{w_n(0)}{w_n((b-a)s_n)} \geq 1 - \frac{1}{w_n(t_0)} \geq \frac{1}{2}$$

since t_0 has been chosen such that $w_n(t_0) \geq 2$. Since $G_n(b-a)$ is a geometric random variable with parameter $p_n^{b-a}(b-a)$, we have

$$\mathbb{E}\left((G_n(b-a))^i\right) \geq \mathbb{E}(G_n(b-a)) = \frac{p_n^{b-a}(b-a)}{1 - p_n^{b-a}(b-a)} \geq 1,$$

using $p_n^{b-a}(b-a) \geq 1/2$. Thus for any $x_0 > 0$,

$$\mathbb{E}\left(|\xi_n^{x_0}|^i\right) = 1 - p_{n,\xi}^{x_0}(a,b) + p_{n,\xi}^{x_0}(a,b)\mathbb{E}\left((G_n(b-a))^i\right)$$
$$\leq (1 - p_{n,\xi}^{x_0}(a,b))\mathbb{E}\left((G_n(b-a))^i\right) + p_{n,\xi}^{x_0}(a,b)\mathbb{E}\left((G_n(b-a))^i\right).$$

This last quantity is equal to $\mathbb{E}\left((G_n(b-a))^i\right)$ and so the inequality $\mathbb{E}\left((G_n(b-a))^i\right)$ $\leq i!(r_n W_n(b-a))^i$ achieves the proof. $\qquad\square$

Lemma 7.15 *For any $i \geq 1$, there exists a finite constant C such that for all $n \geq 1$ and all $0 < a < b$ with $b - a \geq t_0/s_n$,*

$$\mathbb{E}\left(|\xi_n^a|^i\right) \leq C(r_n W_n(b-a))^{i-1}.$$

Moreover, for any $n \geq 1$ and $0 < a < b$,

$$\left|\mathbb{E}(\xi_n^a)\right| \leq \frac{W_n(b-a)}{W_n(a) - W_n(0)}$$

Proof By definition (15) of ξ_n, we have $\mathbb{E}\left(\left|\xi_n^a\right|^i\right) = 1 - p_{n,\xi}^a(a,b) + p_{n,\xi}^a(a,b)\mathbb{E}$ $\left((G_n(b-a))^i\right)$ and so plugging in (24) gives

$$\mathbb{E}\left(\left|\xi_n^a\right|^i\right) \leq 1 + i!\left(\frac{W_n(b-a)}{W_n(0)}\right)^i \frac{W_n(0)(W_n(b) - W_n(b-a))}{W_n(b-a)(W_n(a) - W_n(0))}$$
$$\leq 2i!(r_n W_n(b-a))^{i-1}$$

using $W_n(b) - W_n(b-a) \leq W_n(a) - W_n(0)$ and $1 \leq i!(r_n W_n(b-a))^{i-1}$. The second inequality is a direct consequence of (25) which can be expanded to

$$\left|\mathbb{E}\left(\xi_n^a\right)\right| = \frac{W_n(b-a) - W_n(0) - (W_n(b) - W_n(a))}{W_n(a) - W_n(0)}.$$

The result is proved.   □

Proof of Proposition 7.3 By (17), we have

$$\mathbf{P}_n^*\left(\left|L^0(b) - L^0(a)\right| \geq \lambda \mid T(a) < T(0)\right) = \mathbb{P}\left(\left|\widetilde{\xi}_n^a + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a\right| \geq \lambda r_n\right).$$

We have

$$\mathbb{P}\left(\left|\widetilde{\xi}_n^a + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a\right| \geq \lambda r_n\right)$$
$$\leq C(\lambda r_n)^{-I}\left(\mathbb{E}\left(\left|\widetilde{\xi}_n^a\right|^I\right) + \sum_{\beta \in \mathbb{N}^I:\|\beta\|=I} \mathbb{E}\left((G_n(a))^{|\beta|}\right)\prod_{i=1}^{I}\left|\mathbb{E}\left((\xi_n^a)^i\right)\right|^{\beta_i}\right)$$
$$\leq C\lambda^{-I}\left((W_n(b-a))^I + \sum_{\beta \in \mathbb{N}^I:\|\beta\|=I} r_n^{|\beta|-I}(W_n(a))^{|\beta|}\prod_{i=1}^{I}\left|\mathbb{E}\left((\xi_n^a)^i\right)\right|^{\beta_i}\right)$$

where the first inequality comes from the triangular inequality, Markov inequality and Lemma 7.11, and the second inequality is a consequence of Lemma 7.14 and the fact that $G_n(a)$ is stochastically dominated by an exponential random variable with parameter $1 - p_n^a(a)$. Using Lemma 7.15 and the identity $\sum_{i=2}^{I}(i-1)\beta_i = I - |\beta|$ gives

$$r_n^{|\beta|-I}(W_n(a))^{|\beta|}\prod_{i=1}^{I}\left|\mathbb{E}\left((\xi_n^a)^i\right)\right|^{\beta_i}$$
$$\leq Cr_n^{|\beta|-I}(W_n(a))^{|\beta|}\left(\frac{W_n(b-a)}{W_n(a) - W_n(0)}\right)^{\beta_1}(r_n W_n(b-a))^{I-|\beta|}$$
$$\leq C(W_n(b-a))^{I+\beta_1-|\beta|}.$$

Thus

$$\mathbb{P}\left(\left|\widetilde{\xi}_n^a + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a\right| \geq \lambda r_n\right) \leq C\lambda^{-I}\left((W_n(b-a))^I + \sum_{\beta \in \mathbb{N}^I : \|\beta\|=I} (W_n(b-a))^{I-|\beta|+\beta_1}\right).$$

Since $W_n(t) = w_n(ts_n)/s_n^{\alpha-1}$, it holds that

$$\sup\left\{\frac{W_n(t)}{t^{\alpha-1}} : n \geq 1, t \geq t_0/s_n\right\} = \sup\left\{\frac{w_n(t)}{t^{\alpha-1}} : n \geq 1, t \geq t_0\right\}$$

which has been shown to be finite in the proof of Lemma 7.1. Hence, the last upper bound yields

$$\mathbb{P}\left(\left|\widetilde{\xi}_n^a + \sum_{k=1}^{G_n(a)} \xi_{n,k}^a\right| \geq \lambda r_n\right) \leq C\lambda^{-I}\left((b-a)^{I(\alpha-1)} + \sum_{\beta \in \mathbb{N}^I : \|\beta\|=I} (b-a)^{(I-|\beta|+\beta_1)(\alpha-1)}\right).$$

By (29), $I - |\beta| + \beta_1 \geq I/2$ and since we consider $b - a \leq A$, this gives

$$(b-a)^{(I-|\beta|+\beta_1)(\alpha-1)} \leq C(b-a)^{(\alpha-1)I/2}$$

and we finally get the desired bound for I large enough, i.e., such that $I(\alpha-1) \geq 3$. Inspecting the proof of Proposition 7.2 one can check that one can choose the two constants γ to be equal. $\qquad\square$

References

1. Barlow, M.T.: Necessary and sufficient conditions for the continuity of local time of Lévy processes. Ann. Probab. **16**(4), 1389–1427 (1988)
2. Bertoin, J.: Lévy Processes, volume 121 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge (1996)
3. Bertoin, J.: Exponential decay and ergodicity of completely asymmetric Lévy processes in a finite interval. Ann. Appl. Probab. **7**(1), 156–169 (1997)
4. Billingsley, P. (1999) Convergence of Probability Measures. Wiley Series in Probability and Statistics: Probability and Statistics, 2nd edn. Wiley, New York
5. Borodin, A.N.: The asymptotic behavior of local times of recurrent random walks with finite variance. Teor. Veroyatnost. i Primenen. **26**(4), 769–783 (1981)
6. Borodin, A.N.: Asymptotic behavior of local times of recurrent random walks with infinite variance. Teor. Veroyatnost. i Primenen. **29**(2), 312–326 (1984)
7. Borodin, A.N.: On the character of convergence to Brownian local time. I, II. Probab. Theory Relat. Fields **72**(2), 231–250, 251–277 (1986)
8. Caballero, M.E., Lambert, A., Uribe Bravo, G.: Proof(s) of the Lamperti representation of continuous-state branching processes. Probab. Surv. **6**, 62–89 (2009)
9. Chan, T., Kyprianou, A., Savov, M.: Smoothness of scale functions for spectrally negative Lévy processes. Probab. Theory Relat. Fields, 1–18 (2010). doi:10.1007/s00440-010-0289-4
10. Csáki, E., Révész, P.: Strong invariance for local times. Z. Wahrsch. Verw. Gebiete **62**(2), 263–278 (1983)
11. Csörgő, M., Révész, P.: On strong invariance for local time of partial sums. Stoch. Process. Appl. **20**(1), 59–84 (1985)

12. Duquesne, T., Le Gall, J.-F.: Random trees, Lévy processes and spatial branching processes. Astérisque **281**, vi+147 (2002)
13. Eisenbaum, N., Kaspi, H.: A necessary and sufficient condition for the Markov property of the local time process. Ann. Probab. **21**(3), 1591–1598 (1993)
14. Feller, W.: An Introduction to Probability Theory and Its Applications, vol. II, 2nd edn. Wiley, New York (1971)
15. Grimvall, A.: On the convergence of sequences of branching processes. Ann. Probab. **2**(6), 1027–1045 (1974)
16. Haccou, P., Jagers, P., Vatutin, V.A.: Branching Processes: Variation, Growth, and Extinction of Populations. Cambridge Studies in Adaptive Dynamics. Cambridge University Press, Cambridge (2007)
17. Helland, I.S.: Continuity of a class of random time transformations. Stoch. Process. Appl. **7**(1), 79–99 (1978)
18. Jacod, J., Shiryaev, A.N.: Limit theorems for Stochastic Processes, volume 288 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 2nd edn. Springer, Berlin (2003)
19. Jain, N.C., Pruitt, W.E.: An invariance principle for the local time of a recurrent random walk. Z. Wahrsch. Verw. Gebiete **66**(1), 141–156 (1984)
20. Kang, J.-S., Wee, I.-S.: A note on the weak invariance principle for local times. Stat. Probab. Lett. **32**(2), 147–159 (1997)
21. Kella, O., Zwart, B., Boxma, O.: Some time-dependent properties of symmetric $M/G/1$ queues. J. Appl. Probab. **42**(1), 223–234 (2005)
22. Kesten, H.: An iterated logarithm law for local time. Duke Math. J. **32**, 447–456 (1965)
23. Khoshnevisan, D.: An embedding of compensated compound Poisson processes with applications to local times. Ann. Probab. **21**(1), 340–361 (1993)
24. Knight, F.B.: Random walks and a sojourn density process of Brownian motion. Trans. Am. Math. Soc. **109**, 56–86 (1963)
25. Kuznetsov, A., Kyprianou, A.E., Rivero, V.: The theory of scale functions for spectrally negative Lévy processes. In: Lévy Matters II. Lecture Notes in Mathematics, pp. 97–186. Springer, Berlin, Heidelberg (2013)
26. Kyprianou, A., Rivero, V., Song, R.: Convexity and smoothness of scale functions and de Finetti's control problem. J. Theor. Probab. **23**, 547–564 (2010). doi:10.1007/s10959-009-0220-z
27. Lambert, A.: The contour of splitting trees is a Lévy process. Ann. Probab. **38**(1), 348–395 (2010)
28. Lambert, A., Simatos, F.: The weak convergence of regenerative processes using some excursion path decompositions. Ann. Inst. Henri Poincaré B: Probab. Stat. (accepted)
29. Lambert, A., Simatos, F., Zwart, B.: Scaling limits via excursion theory: interplay between Crump–Mode–Jagers branching processes and processor-sharing queues. Ann. Appl. Probab (accepted)
30. Lamperti, J.: Continuous-state branching processes. Bull. Am. Math. Soc. **73**, 382–386 (1967)
31. Lamperti, J.: The limit of a sequence of branching processes. Probab. Theory Relat. Fields **7**(4), 271–288 (1967)
32. Limic, V.: A LIFO queue in heavy traffic. Ann. Appl. Probab. **11**(2), 301–331 (2001)
33. Perkins, E.: Weak invariance principles for local time. Z. Wahrsch. Verw. Gebiete **60**(4), 437–451 (1982)
34. Révész, P.: Local time and invariance. In: Analytical methods in probability theory (Oberwolfach, 1980), volume 861 of Lecture Notes in Mathematics, pp. 128–145. Springer, Berlin (1981)
35. Révész, P.: A strong invariance principle of the local time of RVs with continuous distribution. Studia Sci. Math. Hungar. **16**(1–2), 219–228 (1981)
36. Robert, P.: Stochastic Networks and Queues. Stochastic Modelling and Applied Probability Series. Springer, New York, xvii+398 pp. (2003)
37. Sagitov, S.M.: General branching processes: convergence to Irzhina processes. J. Math. Sci. **69**(4), 1199–1206 (1994). Stability problems for stochastic models (Kirillov, 1989)
38. Sagitov, S.: A key limit theorem for critical branching processes. Stoch. Process. Appl. **56**(1), 87–100 (1995)
39. Stone, C.: Limit theorems for random walks, birth and death processes, and diffusion processes. Ill. J. Math. **7**, 638–660 (1963)